A CLASS OF NON-SYMMETRIC BAND DETERMINANTS WITH THE GAUSSIAN q-BINOMIAL COEFFICIENTS

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Abstract. A class of symmetric band matrices of bandwidth $2r+1$ with the binomial coefficients entries was studied in [5]. We consider a class of non-symmetric band matrices with the Gaussian q-binomial coefficients whose upper bandwith is s and lower bandwith is r. We give explicit formulae for determinant, inverse and LU-decomposition of the class. We compute the value of infinity-norm of the inverse matrix $\mathcal{H}^{-1}$ for the case $q \to 1$. We use the q-Zeilberger algorithm and unimodality property to prove claimed results.

1. Introduction

There are various combinatorial matrices whose entries are defined in terms of some special integer sequences such as the binomial coefficients, the Fibonacci numbers and the natural numbers. For examples, Pascal matrices are defined via the binomial coefficients (see [4, 3]) and the Hilbert matrix $H = [h_{ij}]$ is defined by the reciprocal of natural numbers as shown

$$h_{ij} = \frac{1}{i+j-1}.$$

As an analogue of the Hilbert matrix, Richardson [16] defined the Filbert matrix $F = [f_{ij}]$ with

$$f_{ij} = \frac{1}{F_{i+j-1}},$$

where F_n is nth Fibonacci number. Recently, Kılıç and Prodinger gave various generalizations and variants of the Filbert matrix (see [6, 7, 8, 9]). For example, they introduced and studied the matrix Q with entries

$$Q_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1}\dots F_{i+j+r+k-1}},$$

where $r \geq -1$ and $k \geq 1$ are integer parameters.

On the other hand, band matrices and their special cases such as Toeplitz matrices, symmetric Toeplitz matrices, especially tridiagonal matrices have been studied by many authors [5, 10, 12, 13].

In 1972, for $r \geq 0$ and for $1 \leq i, j \leq n$, the authors of [5] defined n by n symmetric band matrix $A_n = [a_{ij}]$ of bandwidth $2r+1$ via the binomial coefficients as

$$a_{ij} = (-1)^{r+i-j}\binom{2r}{r+i-j}.$$

For example, when $r = 3$ and $n = 7$,

$$A_7 = \begin{bmatrix} -20 & 15 & -6 & 1 & 0 & 0 & 0 \\ 15 & -20 & 15 & -6 & 1 & 0 & 0 \\ -6 & 15 & -20 & 15 & -6 & 1 & 0 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ 0 & 1 & -6 & 15 & -20 & 15 & -6 \\ 0 & 0 & 1 & -6 & 15 & -20 & 15 \\ 0 & 0 & 0 & 1 & -6 & 15 & -20 \end{bmatrix}.$$

2010 *Mathematics Subject Classification.* 15B36, 15A15, 15A23, 11B65.

Key words and phrases. Determinant, Inverse matrix, LU factorization, Gaussian q-binomial coefficient, Fibonomial coefficients, Zeilberger's algorithm.

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The authors gave formulae for determinant, inverse matrix and LU factorization for the matrix A_n. For example, we have that

$$\det A_n = (-1)^{n+r-1} \prod_{k=1}^{n} \binom{2r+k-1}{r} \binom{k+r-1}{r}^{-1},$$

and

$$\begin{aligned}\left(A_n^{-1}\right)_{ij} &= (-1)^r \binom{i+r-1}{r} \binom{j+r-1}{r} \\ &\times \sum_{k=1}^{n} \binom{k+r-1-i}{r-1} \binom{k+r-1-j}{r} \binom{k+r-1}{r}^{-1} \binom{k+2r-1}{r}^{-1}.\end{aligned}$$

The authors only consider the symmetric band matrix A_n with upper bandwidth of r and lower bandwidth of r.

We should note that a non-symmetric band matrix with upper bandwidth of s and lower bandwidth of r has not been considered and studied up to now. We will consider non-symmetric band matrix with the Gaussian q-binomial coefficients.

The Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k \, (q;q)_{n-k}},$$

where $(x;q)_n$ is the q-Pochhammer symbol, $(x;q)_n = (1-x)(1-xq)\ldots\left(1-xq^{n-1}\right)$.

Note that

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

where $\binom{n}{k}$ is the usual binomial coefficient.

Define the second order linear recurrence $\{U_n\}$ by for $n > 1$,

$$U_n = pU_{n-1} + U_{n-2},$$

with initials $U_0 = 0$ and $U_1 = 1$. The Binet formula is

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1-q^n}{1-q},$$

where $\alpha, \beta = \left(p \mp \sqrt{p^2+4}\right)/2$ and $q = \beta/\alpha$.

For integers n and k such that $n \geq k \geq 0$, the generalized Fibonomial coefficients are defined by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U := \frac{U_1 U_2 \ldots U_n}{(U_1 U_2 \ldots U_k)(U_1 U_2 \ldots U_{n-k})},$$

where $\left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_U = \left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_U = 1$ and 0 otherwise.

When $p = 1$, $U_n = F_n$ (nth Fibonacci number) and $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_U = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_F$ (the usual Fibonomial coefficient).

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \text{ with } q = \beta/\alpha = -\alpha^{-2} \text{ or } \alpha = \mathbf{i} q^{-1/2}.$$

A unimodal sequence is a finite sequence which first increases and then decreases. That is, a sequence $\{a_1, a_2, \ldots, a_n\}$ is unimodal if there exists an integer $t \in \{2, 3, \ldots, n-1\}$ such that

$$a_1 \leq a_2 \leq \cdots \leq a_t \text{ and } a_t \geq a_{t+1} \geq \cdots \geq a_n.$$

In [17], authors studied unimodality of the binomial sequence $\left\{ \binom{n-r}{r} \right\}$. Belbachir and Szalay [1] studied of certain unimodal sequences related to binomial coefficients and they also showed that any sequence laying along a finite ray in any Pascal-pyramid is unimodal in [2].

In this study, we define and study non-symmetric band matrices with upper bandwidth of s and lower bandwidth of r whose entries are defined by the q-binomial coefficients to obtain generalizations of the results of [5]. The case $s = r$ gives us q-analogue of the result of [5]. When $s = r + 1$, we would have a band matrix which has even number of bands, which is not mentioned before.

Briefly we summarize what we done in the paper:

(1) We define the matrix $\mathcal{H}$ with bandwidth $r + s + 1$ via the Gaussian q-binomial coefficients.
(2) We derive explicit formulae for LU factorization and determinant for the matrix $\mathcal{H}$ as well as LU factorization of the matrix $\mathcal{H}^{-1}$.
(3) We derive some complementary results for the work [5] related with the case bandwidth $r + s + 1$.
(4) We compute the value of infinity-norm of the inverse matrix $\mathcal{H}^{-1}$ for the case $q \to 1$.

To prove some claimed results, our main tool is to guess relevant quantities and then use the q-version of Zeilberger's celebrated algorithm (for details, see [14, 15]) to prove them. In the next section, we list our results and then give related proofs in the following section. All identities we will obtain hold for general q, and results about Fibonomial coefficients come out as corollaries for the special choice of q.

For each section, in general the size of the matrix does not really matter except the results about inverse matrix, so that we may think about an infinite matrix M and restrict it whenever necessary to the first n rows resp. columns and use the notation M_n.

Throughout the paper, we use the letters L, U and A, B for LU factorizations of the matrix $\mathcal{H}$ and its inverse, respectively. For LU factorizations of the matrices $\mathcal{C}$ and $\mathcal{C}^{-1}$, we use the calligraphic letters $\mathcal{L}$, $\mathcal{U}$, $\mathcal{A}$ and $\mathcal{B}$. That is, $\mathcal{H} = LU$, $\mathcal{H}^{-1} = AB$, $\mathcal{C} = \mathcal{LU}$ and $\mathcal{C}^{-1} = \mathcal{AB}$.

2. Band Matrices of Bandwidth $r + s + 1$

Define the matrix $\mathcal{H} = [h_{kj}]$ with upper bandwidth s and lower bandwidth r with

$$h_{k,j} = (-1)^{r(k+j)+j}\, \mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)} q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs} \begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q,$$

for $0 \le k, j \le n-1$ and nonnegative arbitrary integers r and s where $\mathbf{i} = \sqrt{-1}$.

For example, when $r = 2$ and $s = 4$, we have

$$\mathcal{H} = \begin{bmatrix}
-q^{-4}\begin{bmatrix}6\\2\end{bmatrix}_q & -\mathbf{i}q^{-\frac{9}{2}}\begin{bmatrix}6\\3\end{bmatrix}_q & q^{-4}\begin{bmatrix}6\\4\end{bmatrix}_q & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\5\end{bmatrix}_q & -1 & & 0 \\
\mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\1\end{bmatrix}_q & -q^{-4}\begin{bmatrix}6\\2\end{bmatrix}_q & -\mathbf{i}q^{-\frac{9}{2}}\begin{bmatrix}6\\3\end{bmatrix}_q & q^{-4}\begin{bmatrix}6\\4\end{bmatrix}_q & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\5\end{bmatrix}_q & \ddots & \\
1 & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\1\end{bmatrix}_q & -q^{-4}\begin{bmatrix}6\\2\end{bmatrix}_q & \ddots & \ddots & \ddots & -1 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\5\end{bmatrix}_q \\
 & & \ddots & \ddots & \ddots & \ddots & q^{-4}\begin{bmatrix}6\\4\end{bmatrix}_q \\
 & & & \ddots & \ddots & \ddots & -\mathbf{i}q^{-\frac{9}{2}}\begin{bmatrix}6\\3\end{bmatrix}_q \\
0 & & & & 1 & \mathbf{i}q^{-\frac{5}{2}}\begin{bmatrix}6\\1\end{bmatrix}_q & -q^{-4}\begin{bmatrix}6\\2\end{bmatrix}_q
\end{bmatrix}.$$

When we take $q = \beta/\alpha$, we denote Fibonomial analogue of $\mathcal{H}$ by $\mathcal{C} = [c_{kj}]$. For $0 \le k, j \le n-1$,

$$c_{k,j} = (-1)^{r(k+j)+j(k+1)}\, \mathbf{i}^{j(j+1)+k(k+1)+r(r-1)} \left\{ {r+s \atop r+j-k} \right\}_U.$$

For $r = 2$ and $s = 4$,

$$\mathcal{C} = \begin{bmatrix}
-\left\{{6 \atop 2}\right\}_U & -\left\{{6 \atop 3}\right\}_U & \left\{{6 \atop 4}\right\}_U & \left\{{6 \atop 5}\right\}_U & -1 & & 0 \\
\left\{{6 \atop 1}\right\}_U & -\left\{{6 \atop 2}\right\}_U & -\left\{{6 \atop 3}\right\}_U & \left\{{6 \atop 4}\right\}_U & \left\{{6 \atop 5}\right\}_U & \ddots & \\
1 & \left\{{6 \atop 1}\right\}_U & -\left\{{6 \atop 2}\right\}_U & \ddots & \ddots & \ddots & -1 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \left\{{6 \atop 5}\right\}_U \\
 & & \ddots & \ddots & \ddots & \ddots & \left\{{6 \atop 4}\right\}_U \\
 & & & \ddots & \ddots & \ddots & -\left\{{6 \atop 3}\right\}_U \\
0 & & & & 1 & \left\{{6 \atop 1}\right\}_U & -\left\{{6 \atop 2}\right\}_U
\end{bmatrix}.$$

We obtain the LU decomposition $\mathcal{H} = L \cdot U$:

Theorem 1. *For* $0 \le k, j \le n-1$

$$L_{kj} = (-1)^{r(k+j)+j}\, \mathbf{i}^{k+j+(r-s)(k-j)} q^{\frac{1}{2}(k-j)(k-j+s-r)} \begin{bmatrix} r \\ k-j \end{bmatrix}_q \begin{bmatrix} k \\ k-j \end{bmatrix}_q \begin{bmatrix} s+k \\ k-j \end{bmatrix}_q^{-1}.$$

Theorem 2. *For* $0 \le k, j \le n-1$,

$$U_{kj} = (-1)^{r(k+j)+j}\, \mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)} q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs} \times \begin{bmatrix} s \\ j-k \end{bmatrix}_q \begin{bmatrix} r+s+k \\ r+j \end{bmatrix}_q \begin{bmatrix} s+k \\ j \end{bmatrix}_q^{-1}.$$

As Fibonomial corollaries of Theorems 1 and 2, we get

Corollary 1. *For* $0 \le k, j \le n-1$,

$$\mathcal{L}_{kj} = (-1)^{r(k+j)+j(k+1)}\, \mathbf{i}^{k(k+1)+j(j+1)} \begin{Bmatrix} r \\ k-j \end{Bmatrix}_U \begin{Bmatrix} k \\ k-j \end{Bmatrix}_U \begin{Bmatrix} s+k \\ k-j \end{Bmatrix}_U^{-1}.$$

Corollary 2. *For* $0 \le k, j \le n-1$,

$$\mathcal{U}_{kj} = (-1)^{r(k+j)+j(k+1)}\, \mathbf{i}^{j(j+1)+k(k+1)+r(r-1)} \begin{Bmatrix} s \\ j-k \end{Bmatrix}_U \begin{Bmatrix} r+s+k \\ r+j \end{Bmatrix}_U \begin{Bmatrix} s+k \\ j \end{Bmatrix}_U^{-1}.$$

We give inverse matrices L^{-1} and U^{-1} by the following two theorems.

Theorem 3. *For* $0 \le k, j \le n-1$,

$$L^{-1}_{kj} = (-1)^{r(k+j)}\, \mathbf{i}^{(k-j)(r-1-s)} q^{\frac{1}{2}(k-j)(s-r+1)} \begin{bmatrix} k-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} s+j \\ j \end{bmatrix}_q \begin{bmatrix} s+k \\ k \end{bmatrix}_q^{-1}.$$

Its Fibonomial corollary is

Corollary 3. *For* $0 \le k, j \le n-1$,

$$\mathcal{L}^{-1}_{kj} = (-1)^{(k+j)r} \begin{Bmatrix} k-j+r-1 \\ r-1 \end{Bmatrix}_U \begin{Bmatrix} s+j \\ j \end{Bmatrix}_U \begin{Bmatrix} s+k \\ k \end{Bmatrix}_U^{-1}.$$

Theorem 4. *For* $0 \le k, j \le n-1$,

$$U^{-1}_{kj} = (-1)^{(k+j)(r+1)}\, \mathbf{i}^{k-j-r+(k+r-j)(r-s)} q^{\frac{1}{2}(k-j)(s-r-1)+\frac{1}{2}rs} \times \begin{bmatrix} j-k+s-1 \\ s-1 \end{bmatrix}_q \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} r+s+j \\ r \end{bmatrix}_q^{-1}.$$

Its Fibonomial corollary is

Corollary 4. *For* $0 \le k, j \le n-1$,

$$\mathcal{U}^{-1}_{kj} = (-1)^{(j+k)(r+1)}\, \mathbf{i}^{r(r-1)} \begin{Bmatrix} j-k+s-1 \\ s-1 \end{Bmatrix}_U \begin{Bmatrix} r+k \\ k \end{Bmatrix}_U \begin{Bmatrix} r+s+j \\ r \end{Bmatrix}_U^{-1}.$$

As a consequence, we may give determinants of $\mathcal{H}$ and $\mathcal{C}$, since it is simply evaluated as products of the main diagonal entries of the upper triangular matrices U and $\mathcal{U}$ (we only state the Fibonomial version):

Theorem 5.

$$\det \mathcal{C}_n = \mathbf{i}^{nr(r-1)} \prod_{m=0}^{n-1} \begin{Bmatrix} r+s+m \\ r+m \end{Bmatrix}_U \begin{Bmatrix} s+m \\ m \end{Bmatrix}_U^{-1}.$$

Now we compute inverse of the matrix $\mathcal{H}_n$ as follows:

Theorem 6. *For* $0 \le k, j \le n-1$,

$$h_{k,j}^{-1} = (-1)^{r(j+k)+k}\, \mathbf{i}^{k(1+r-s)+j(1-r+s)+r(r-s-1)} q^{\frac{1}{2}((k-j)(s-r)+rs-j-k)} \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} s+j \\ j \end{bmatrix}_q$$
$$\times \sum_{m=0}^{n-1} q^m \begin{bmatrix} m-k+s-1 \\ s-1 \end{bmatrix}_q \begin{bmatrix} m-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} r+s+m \\ r \end{bmatrix}_q^{-1} \begin{bmatrix} s+m \\ m \end{bmatrix}_q^{-1}.$$

So we give the LU-factorization of $\mathcal{H}_n^{-1}$, that is, $\mathcal{H}_n^{-1} = AB$ and also find inverses of these factors.

Theorem 7. *For* $0 \le k, j \le n-1$,

$$A_{kj} = (-1)^{r(k+j)}\, \mathbf{i}^{(k-j)(r-1-s)} q^{\frac{1}{2}(k-j)(s-r+1)} \times \begin{bmatrix} k-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} s+n-1-k \\ s \end{bmatrix}_q \begin{bmatrix} s+n-1-j \\ s \end{bmatrix}_q^{-1}$$

and for $0 \le k, j \le n-1$,

$$A_{kj}^{-1} = (-1)^{r(k+j)+j}\, \mathbf{i}^{k+j+(r-s)(k-j)} q^{\frac{1}{2}(k-j)(k-j+s-r)} \times \begin{bmatrix} r \\ k-j \end{bmatrix}_q \begin{bmatrix} n-1-j \\ k-j \end{bmatrix}_q \begin{bmatrix} s+n-1-j \\ k-j \end{bmatrix}_q^{-1}.$$

Its Fibonomial corollary is

Corollary 5. *For* $0 \le k, j \le n-1$,

$$\mathcal{A}_{kj} = (-1)^{(k+j)r} \begin{Bmatrix} k-j+r-1 \\ r-1 \end{Bmatrix}_U \begin{Bmatrix} s+n-1-k \\ s \end{Bmatrix}_U \begin{Bmatrix} s+n-1-j \\ s \end{Bmatrix}_U^{-1}$$

and for $0 \le k, j \le n-1$,

$$\mathcal{A}_{kj}^{-1} = (-1)^{r(k+j)+j(k+1)}\, \mathbf{i}^{k(k+1)+j(j+1)} \times \begin{Bmatrix} r \\ k-j \end{Bmatrix}_U \begin{Bmatrix} n-1-j \\ k-j \end{Bmatrix}_U \begin{Bmatrix} s+n-1-j \\ k-j \end{Bmatrix}_U^{-1}.$$

Theorem 8. *For* $0 \le k, j \le n-1$,

$$B_{kj} = (-1)^{(k+j)(r+1)}\, \mathbf{i}^{k-j-r+(k+r-j)(r-s)} q^{\frac{1}{2}(k-j)(s-r-1)+\frac{1}{2}rs} \times \begin{bmatrix} j-k+s-1 \\ s-1 \end{bmatrix}_q \begin{bmatrix} r+n-1-j \\ r \end{bmatrix}_q \begin{bmatrix} r+s+n-1-k \\ r \end{bmatrix}_q^{-1}$$

and for $0 \le k, j \le n-1$,

$$B_{kj}^{-1} = (-1)^{r(k+j)+j}\, \mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)} q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs} \times \begin{bmatrix} s \\ j-k \end{bmatrix}_q \begin{bmatrix} r+s+n-1-j \\ s+k-j \end{bmatrix}_q \begin{bmatrix} s+n-1-j \\ s+k-j \end{bmatrix}_q^{-1}.$$

Its Fibonomial corollary is

Corollary 6. *For* $0 \le k, j \le n-1$,

$$\mathcal{B}_{kj} = (-1)^{(j+k)(r+1)}\, \mathbf{i}^{r(r-1)} \begin{Bmatrix} j-k+s-1 \\ s-1 \end{Bmatrix}_U \times \begin{Bmatrix} r+n-1-j \\ r \end{Bmatrix}_U \begin{Bmatrix} r+s+n-1-k \\ r \end{Bmatrix}_U^{-1}$$

and for $0 \le k, j \le n-1$,

$$\mathcal{B}_{kj}^{-1} = (-1)^{r(k+j)+j(k+1)}\, \mathbf{i}^{j(j+1)+k(k+1)+r(r-1)} \times \begin{Bmatrix} s \\ j-k \end{Bmatrix}_U \begin{Bmatrix} r+s+n-1-j \\ s+k-j \end{Bmatrix}_U \begin{Bmatrix} s+n-1-j \\ s+k-j \end{Bmatrix}_U^{-1}.$$

3. Proofs

In order to show that indeed $\mathcal{H} = L \cdot U$, it is sufficient to show that

$$\sum_{0 \le d \le \min\{k,j\}} L_{kd} U_{dj} = h_{kj}.$$

We have

$$\begin{aligned}
&\sum_{0\le d\le \min\{k,j\}} (-1)^{r(k+d)+d}\, \mathbf{i}^{(d+k)+(r-s)(k-d)} q^{\frac{1}{2}(k-d)(k-d+s-r)} \begin{bmatrix} r \\ k-d \end{bmatrix}_q \begin{bmatrix} k \\ k-d \end{bmatrix}_q \\
&\times \begin{bmatrix} s+k \\ k-d \end{bmatrix}_q^{-1} (-1)^{r(d+j)+j}\, \mathbf{i}^{d(1+r-s)+j(1-r+s)-r(1-s-r)} \\
&\times q^{\frac{1}{2}(d-j)(d-j-r+s)-\frac{1}{2}rs} \begin{bmatrix} s \\ j-d \end{bmatrix}_q \begin{bmatrix} r+s+d \\ r+j \end{bmatrix}_q \begin{bmatrix} s+d \\ j \end{bmatrix}_q^{-1} \\
&= (-1)^{r(k+j)+j}\, \mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)} q^{\frac{1}{2}(k-j)(k-j-r+s)-\frac{1}{2}rs} \begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q .
\end{aligned}$$

Rewrite the equality just above as

$$\begin{aligned}
&\sum_{j-s\le d\le k} q^{-d(k+j)+d^2} \begin{bmatrix} r \\ k-d \end{bmatrix}_q \begin{bmatrix} k \\ k-d \end{bmatrix}_q \begin{bmatrix} s+k \\ k-d \end{bmatrix}_q^{-1} \begin{bmatrix} s \\ j-d \end{bmatrix}_q \begin{bmatrix} r+s+d \\ r+j \end{bmatrix}_q \begin{bmatrix} s+d \\ j \end{bmatrix}_q^{-1} \\
&= q^{-jk} \begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q .
\end{aligned}$$

Denote the LHS of the last equation by $\mathtt{SUM}_k$. The Mathematica version of the q-Zeilberger algorithm [14] produces the recursion

$$\mathtt{SUM}_k = \frac{q^{-j}\left(1-q^{1+j-k+r}\right)}{\left(1-q^{-j+k+s}\right)} \mathtt{SUM}_{k-1}.$$

Solving the recurrence, we obtain

$$\mathtt{SUM}_k = \frac{q^{-jk}\left(1-q^{1+j-k+r}\right)\ldots\left(1-q^{r+j}\right)}{\left(1-q^{s-j+k}\right)\ldots\left(1-q^{s-j+1}\right)} \mathtt{SUM}_0,$$

where $\mathtt{SUM}_0 = \begin{bmatrix} r+s \\ r+j \end{bmatrix}_q$. After multiplying both the denominator and numerator of the above equation with $(q;q)_{j-k+r}$, we obtain

$$\mathtt{SUM}_k = q^{-jk} \begin{bmatrix} r+s \\ r+j-k \end{bmatrix}_q ,$$

as claimed.

Now look at the inverse matrices. Since L and L^{-1} are lower triangular matrices, we only need to look the entries indexed by (k,j) with $k \ge j$. So we must show that

$$\sum_{j \le d \le k} L_{kd} L^{-1}_{dj} = \delta_{kj},$$

where δ_{kj} is Kronecker delta.

Then we obtain

$$\begin{aligned}
\sum_{j\le d\le k} L_{kd} L^{-1}_{dj} &= (-1)^{r(k+j)}\, \mathbf{i}^{k+j+(k-j)(r-s)} q^{\frac{1}{2}\left(k^2-j\right)+\frac{1}{2}(r-s)(j-k)} \begin{bmatrix} s+j \\ j \end{bmatrix}_q \\
&\times \sum_{j\le d\le k} (-1)^d\, q^{\frac{1}{2}\left(d^2+d\right)-kd} \begin{bmatrix} r \\ k-d \end{bmatrix}_q \begin{bmatrix} k \\ d \end{bmatrix}_q \begin{bmatrix} s+k \\ k-d \end{bmatrix}_q^{-1} \begin{bmatrix} d-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} s+d \\ d \end{bmatrix}_q^{-1} .
\end{aligned}$$

The q-Zeilberger algorithm evaluates the sum on the RHS of the last equation as 0 when $k \ne j$ and $r \ne 0$. For the case $r = 0$, we have an upper triangular matrix so that it is clear. For the case $k = j$, it is easy to check that $L_{kk} L^{-1}_{kk} = 1$. So proof is completed.

Since U and U^{-1} are upper triangular matrices, we just need to look the entries indexed by (k,j) with $j \geq k$. Thus we have the sum

$$\sum_{k\leq d\leq j} U_{kd}U^{-1}_{dj} = (-1)^{r(j+k+r-1)+j}\,\mathbf{i}^{(k-j)(r-s+1)} q^{\frac{1}{2}\left((r-s)(j-k)+k^2+j\right)} \begin{bmatrix} r+s+j \\ r \end{bmatrix}_q^{-1}$$
$$\times \sum_{k\leq d\leq j} (-1)^d q^{\frac{1}{2}\left(d^2-d\right)-kd} \begin{bmatrix} s \\ d-k \end{bmatrix}_q \begin{bmatrix} r+s+k \\ s-d+k \end{bmatrix}_q$$
$$\times \begin{bmatrix} s+k \\ d \end{bmatrix}_q^{-1} \begin{bmatrix} j-d+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} r+d \\ r \end{bmatrix}_q.$$

The q-Zeilberger algorithm evaluates the sum on the RHS of the last equation as 0 when $k \neq j$ and $s \neq 0$. When we choose the number of super-diagonals of the matrix $\mathcal{H}$ as zero, that is $s = 0$, it is easy to check because the matrix $\mathcal{H}$ is a lower triangular matrix. If $k = j$, it is obvious that $U_{kk}U^{-1}_{kk} = 1$. Thus

$$\sum_{k\leq d\leq j} U_{kd}U^{-1}_{dj} = \delta_{kj},$$

as claimed.

Now we turn to the inverse matrix $\mathcal{H}_n^{-1}$. Using the fact that $\mathcal{H}_n^{-1} = U_n^{-1}L_n^{-1}$, we can write

$$h^{-1}_{kj} = \sum_{d=0}^{n-1} U^{-1}_{kd}L^{-1}_{dj} = \sum_{d=0}^{n-1} (-1)^{(k+d)(r+1)}\,\mathbf{i}^{(k-d-r)+(k+r-d)(r-s)} q^{\frac{1}{2}((k-d)(s-r-1)+rs)}$$
$$\times \begin{bmatrix} d-k+s-1 \\ s-1 \end{bmatrix}_q \begin{bmatrix} r+k \\ k \end{bmatrix}_q \begin{bmatrix} r+s+d \\ r \end{bmatrix}_q^{-1} (-1)^{r(d+j)}\,\mathbf{i}^{(d-j)(r-1-s)}$$
$$\times\, q^{\frac{1}{2}(d-j)(s-r+1)} \begin{bmatrix} d-j+r-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} s+j \\ j \end{bmatrix}_q \begin{bmatrix} s+d \\ d \end{bmatrix}_q^{-1}.$$

After some straightforward simplifications, we obtain Theorem 6. Unfortunately, this sum cannot be evaluated in closed form.

For the verification of the LU-factorization of inverse matrix, we give the following Lemma. Recall that the letters L, U, and, A, B are used for LU factorizations of a matrix and its inverse, respectively.

Lemma 1. *If M is a Toeplitz matrix of order n. There exist the following relationships between the factor matrices come from LU-decompositions of the matrices M and M^{-1} : for $0 \leq k, j \leq n-1$,*

(i) $A_{kj} = L^{-1}_{n-1-j,n-1-k}$,
(ii) $A^{-1}_{kj} = L_{n-1-j,n-1-k}$,
(iii) $B_{kj} = U^{-1}_{n-1-j,n-1-k}$,
(iv) $B^{-1}_{kj} = U_{n-1-j,n-1-k}$,
(v) $M^{-1}_{kj} = M^{-1}_{n-1-j,n-1-k}$.

Proof. For the first two claims, consider

$$\sum_{d=j}^{k} A_{kd}A^{-1}_{dj} = \sum_{d=j}^{k} L^{-1}_{n-1-d,n-1-k}L_{n-1-j,n-1-d} = \sum_{d=n-1-k}^{n-1-j} L^{-1}_{d,n-1-k}L_{n-1-j,d} = \delta_{n-1-j,n-1-k},$$

gives us $AA^{-1} = I$, as claimed. The claims (iii) and (iv) can be similarly done.

For the LU-decomposition of M^{-1}, we should show that $M^{-1} = AB$ or equivalently $M = B^{-1}A^{-1}$. So it is sufficient to show that

$$\sum_{\max\{k,j\}\leq d\leq n-1} B^{-1}_{kd}A^{-1}_{dj} = m_{kj}.$$

Thus consider

$$\begin{aligned}\sum_{\max\{k,j\}\le d\le n-1} B^{-1}_{kd}A^{-1}_{dj} &= \sum_{\max\{k,j\}\le d\le n-1} U_{n-1-d,n-1-k}L_{n-1-j,n-1-d}\\ &= \sum_{0\le d\le n-1-\max\{k,j\}} U_{d,n-1-k}L_{n-1-j,d}\\ &= \sum_{0\le d\le \min\{n-1-j,n-1-k\}} L_{n-1-j,d}U_{d,n-1-k}.\end{aligned}$$

Since $M=LU$ and M is a Toeplitz matrix, we have $\sum_{0\le d\le \min\{k,j\}} L_{kd}U_{dj}=m_{kj}$ and $m_{kj}=m_{n-1-j,n-1-k}$. Finally, we obtain

$$\sum_{\max\{k,j\}\le d\le n-1} B^{-1}_{kd}A^{-1}_{dj}=m_{n-1-j,n-1-k}=m_{kj},$$

which completes the proof. The result (v) can be easily derived by the fact that $M^{-1}=U^{-1}L^{-1}=AB$. $\square$

Since the matrix $\mathcal{H}$ is a Toeplitz matrix, Theorems 7 and 8 arise as consequences of Lemma 1.

4. The Case Bandwidth $r+s+1$ with the Binomial Coefficient

For $r\ge 0$, we define the non-symmetrix band matrix $\mathcal{D}=[d_{kj}]$ via the binomial coefficients by for $0\le k,j\le n-1$,

$$d_{kj}=(-1)^{k+j+r}\binom{r+s}{r+j-k}.$$

For example, when $r=2$, $s=4$ and $n=7$, we have:

$$\mathcal{D}_7=\begin{bmatrix} 15 & -20 & 15 & -6 & 1 & 0 & 0\\ -6 & 15 & -20 & 15 & -6 & 1 & 0\\ 1 & -6 & 15 & -20 & 15 & -6 & 1\\ 0 & 1 & -6 & 15 & -20 & 15 & -6\\ 0 & 0 & 1 & -6 & 15 & -20 & 15\\ 0 & 0 & 0 & 1 & -6 & 15 & -20\\ 0 & 0 & 0 & 0 & 1 & -6 & 15 \end{bmatrix}.$$

We list the results related with LU-decomposition, L^{-1}, U^{-1} and determinant of the matrix $\mathcal{D}$, respectively. We give the results without proof not to bore the readers.

Theorem 9. *For $0\le k,j\le n-1$,*

$$L_{kj}=(-1)^{k+j}\binom{r}{k-j}\binom{k}{j}\binom{s+k}{k-j}^{-1}.$$

Theorem 10. *For $0\le k,j\le n-1$,*

$$U_{kj}=(-1)^{k+j+r}\binom{s}{j-k}\binom{r+s+k}{r+j}\binom{s+k}{j}^{-1}.$$

Theorem 11. *For $0\le k,j\le n-1$,*

$$L^{-1}_{kj}=\binom{k-j+r-1}{r-1}\binom{s+j}{j}\binom{s+k}{k}^{-1}.$$

Theorem 12. *For $0\le k,j\le n-1$,*

$$U^{-1}_{kj}=(-1)^{r}\binom{j-k+s-1}{s-1}\binom{r+k}{k}\binom{r+s+j}{r}^{-1}.$$

Theorem 13. *For $n>1$,*

$$\det\mathcal{D}_n=(-1)^{rn}\prod_{m=0}^{n-1}\binom{r+s+m}{r+m}\binom{s+m}{m}^{-1}.$$

Similarly, we have the following result for the LU-factorization of the matrix $\mathcal{D}_n^{-1}$.

Theorem 14. *For $0 \le k, j \le n-1$,*

$$A_{kj} = \binom{k-j+r-1}{r-1}\binom{s+n-1-k}{s}\binom{s+n-1-j}{s}^{-1},$$

$$A_{kj}^{-1} = (-1)^{k+j}\binom{r}{k-j}\binom{n-1-j}{k-j}\binom{s+n-1-j}{k-j}^{-1},$$

$$B_{kj} = (-1)^{r}\binom{j-k+s-1}{s-1}\binom{r+n-1-j}{r}\binom{r+s+n-1-k}{r}^{-1},$$

and

$$B_{kj}^{-1} = (-1)^{k+j+r}\binom{s}{j-k}\binom{r+s+n-1-j}{s+k-j}\binom{s+n-1-j}{s+k-j}^{-1}.$$

Now we present the result for infinity-norm of the matrix $\mathcal{D}_n^{-1}$, which is the maximum value of the absolute row sum, that is,

$$\left\|\mathcal{D}_n^{-1}\right\|_\infty = \max_k \left\{ \sum_{j=0}^{n-1} \left|d_{kj}^{-1}\right|, 0 \le k \le n-1 \right\}.$$

First we give the following lemmas:

Lemma 2. *For $0 \le k \le n-1$, the kth row sum, denoted by S_k, of $\mathcal{D}_n^{-1}$ is*

$$S_k = (-1)^r \frac{\binom{k+r}{r}\binom{n-k-1+s}{s}}{\binom{r+s}{r}}.$$

Proof. Denote the unit vector by e_k, where 1 in the kth position and denote the vector where all position consists of 1 by e. Then we write

$$S_k = e_k^T \mathcal{D}_n^{-1} e.$$

Since there is no closed formula for $\mathcal{D}_n^{-1}$, we will use the result $\mathcal{D}^{-1} = U^{-1}L^{-1}$, where the matrices L^{-1} and U^{-1} are computed in Theorems 11 and 12, resp. Thus we should compute

$$S_k = \left(e_k^T U_n^{-1}\right)\left(L_n^{-1} e\right).$$

Here the second parenthesis gives row sum of the matrix L_n^{-1} and the first parenthesis gives the kth row of the matrix of U_n^{-1}. So the sum of kth row of the matrix L_n^{-1}, denoted by s_k, is

$$s_k = \sum_{j=0}^{k} \binom{k-j+r-1}{r-1}\binom{s+j}{j}\binom{s+k}{k}^{-1},$$

which, by the Vandermonde Convolution equation 5.26 in [11], equals

$$\binom{s+k}{k}^{-1}\binom{k+r+s}{r+s} = \binom{k+r+s}{r}\binom{r+s}{r}^{-1}.$$

Consequently, we have that $\left(L_n^{-1}e\right) = [s_0, s_1, \ldots, s_{n-1}]^T$ and $\left(e_k^T U_n^{-1}\right) = \left[0, 0, \ldots, u_{kk}^{-1}, u_{k,k+1}^{-1}, \ldots, u_{k,n-1}^{-1}\right]$. Finally, we obtain

$$\begin{aligned} S_k = \sum_{j=k}^{n-1} u_{kj}^{-1} s_j &= (-1)^r \binom{r+s}{s}^{-1}\binom{r+k}{r}\sum_{j=k}^{n-1}\binom{j-k+s-1}{s-1} \\ &= (-1)^r \binom{r+s}{s}^{-1}\binom{r+k}{r}\sum_{j=0}^{n-k-1}\binom{j+s-1}{j} \\ &= (-1)^r \binom{r+s}{s}^{-1}\binom{r+k}{r}\binom{s+n-k-1}{s}, \end{aligned}$$

as claimed. $\square$

In order to evaluate the infinity-norm of the matrix $\mathcal{D}_n^{-1}$, we need the maximum value of $|S_k|$. For this, we investigate unimodality of $\{|S_k|\}$.

Lemma 3. *The sequence $\{|S_k|\}$ is unimodal.*

Proof. Since the factor $\binom{r+s}{s}^{-1}$ is independent from the index k, we should show that

$$\{a_k\} = \left\{ \binom{r+k}{r} \binom{s+n-k-1}{s} \right\}$$

is unimodal instead of showing the unimodality of the sequence $\{|S_k|\}$. Consider

$$\begin{aligned}
a_k^2 &= \binom{r+k}{r}^2 \binom{s+n-k-1}{s}^2 \\
&= \frac{(k+r)(k+1)(n-k-1+s)(n-k)}{(k+r+1)k(n-k+s)(n-k-1)} a_{k-1}a_{k+1} \\
&= \left(1 - \frac{1}{k+r+1}\right)\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{n-k+s}\right)\left(1 + \frac{1}{n-k-1}\right) a_{k-1}a_{k+1} \\
&= \left(1 + \frac{r}{k^2+kr+k}\right)\left(1 + \frac{s}{(k-n+1)(k-n-s)}\right) a_{k-1}a_{k+1} \\
&> a_{k-1}a_{k+1},
\end{aligned}$$

which says us that the sequence $\{a_k\}$ is log-concave that means $\{a_k\}$ is unimodal. $\square$

Since the sequence $\{|S_k|\}$ is unimodal, it has maximum value for some k, where $k \in \{0, 1, \dots, n-1\}$. Thus we can compute the $\left\|\mathcal{D}_n^{-1}\right\|_\infty$:

Theorem 15. *For $n > 1$,*

$$\left\|\mathcal{D}_n^{-1}\right\|_\infty = \frac{(r+t+1)^{\underline{r}}(s+n-t)^{\underline{s}}}{(r+s)!},$$

where $t = \left\lfloor \dfrac{nr}{r+s} - 1 \right\rfloor$ and the falling factorial is defined as $x^{\underline{n}} = x(x-1)\dots(x-n+1)$.

Proof. We know that there exist a nonnegative integer $k \in \{0, 1, \dots, n-1\}$ so that $|S_k|$ is the maximum value. We shall find this value of k. We only consider the sequence $\{a_k\} = \left\{ \binom{r+k}{r} \binom{s+n-k-1}{s} \right\}$ instead of the sequence $\{S_k\}$ because it is enough to consider the factors only depend on k. Consider

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\binom{k+1+r}{r}\binom{n-k-2+s}{s}}{\binom{k+r}{r}\binom{n-k-1+s}{s}} \\
&= \frac{(k+1+r)(n-k-1)}{(k+1)(n-k-1+s)} \\
&= \left(1 + \frac{r}{k+1}\right)\left(1 - \frac{s}{n-k-1+s}\right) \\
&= 1 + \frac{nr - (k+1)(s+r)}{(k+1)(n-k-1+s)}.
\end{aligned}$$

If $nr - (k+1)(s+r) > 0$, then $k < \frac{nr}{r+s} - 1$ and so $\{a_k\}$ is increasing for such k's. When $k > \frac{nr}{r+s} - 1$, the sequence $\{a_k\}$ is decreasing. Since k is an integer, the sequence $\{a_k\}$ takes the maximum value at $k = \left\lfloor \frac{nr}{r+s} - 1 \right\rfloor + 1$, which completes the proof. $\square$

Denote the sum of the jth column entries of the matrix $\mathcal{D}_n^{-1}$ by $\overline{S}_j$. By Lemma 1(v), we can see that $S_k = \overline{S}_{n-1-k}$. So we derive the result

$$\left\|\mathcal{D}_n^{-1}\right\|_1 = S_{n-1-t},$$

where $t = \left\lfloor \frac{nr}{r+s} - 1 \right\rfloor$ and $\|\cdot\|_1$ is the maximum absolute column sum norm.

5. Conclusion

If we take $q \to 1$ in our results related with the Gaussian q-binomial coefficients, then we get results for matrices including the usual binomial coefficients. Thus the matrix $\mathcal{H}$ takes the form for $q \to 1$

$$h_{k,j} = (-1)^{r(k+j)+j}\, \mathbf{i}^{k(1+r-s)+j(1-r+s)-r(1-s-r)} \binom{r+s}{r+j-k}.$$

On the other hand, the matrix $\mathcal{D}$ in Section 4 was defined by

$$a_{k,j} = (-1)^{r+k-j} \binom{r+s}{r+j-k}.$$

Although the matrices $\mathcal{H}_{q\to 1}$ and $\mathcal{D}$ seem different from each other since sings of their entries, this does not effect their determinant values since the properties of banded matrices. Note that the signs of the multiplication of the entries on the corresponding mth superdiagonal and mth subdiagonal of these matrices according to the Laplace expansion of determinant are the same.

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