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On a class of nonlocal wave equations from applications

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We study equations from the area of peridynamics, which is a nonlocal extension of elasticity. The governing equations form a system of nonlocal wave equations. We take a novel approach by applying operator theory methods in a systematic way. On the unbounded domain $\mathbb{R}^n$, we present three main results. As main result 1, we find that the governing operator is a bounded function of the governing operator of classical elasticity. As main result 2, a consequence of main result 1, we prove that the peridynamic solutions strongly converge to the classical solutions by utilizing, for the first time, strong resolvent convergence. In addition, main result 1 allows us to incorporate local boundary conditions, in particular, into peridynamics. This avenue of research is developed in companion papers, providing a remedy for boundary effects. As main result 3, employing spherical Bessel functions, we give a new practical series representation of the solution which allows straightforward numerical treatment with symbolic computation. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4953252]

I. INTRODUCTION AND MOTIVATION

Classical elasticity has been successful in characterizing and measuring the resistance of materials to crack growth. On the other hand, peridynamics (PD), a nonlocal extension of continuum mechanics developed by Silling,[55] is capable of quantitatively predicting the dynamics of propagating cracks, including bifurcation. Its effectiveness has been established in sophisticated applications such as Kalthoff-Winkler experiments of the fracture of a steel plate with notches,[31,58] fracture and failure of composites, nanofiber networks, and polycrystal fracture.[34,44,60,59] Further applications are in the context of multiscale modeling, where PD has been shown to be an upscaling of molecular dynamics[52,54] and has been demonstrated as a viable multiscale material model for length scales ranging from molecular dynamics to classical elasticity.[10] Also see other related engineering applications,[13,33,35,46,45] the review, and news articles[14,16,21,57] for a comprehensive discussion, and the book.[38]

We study a class of nonlocal wave equations. The driving application is PD whose equation of motion corresponds exactly to the nonlocal wave equation under consideration. The same operator is also employed in nonlocal diffusion.[9,14,51] Similar classes of operators are used in numerous applications such as population models, particle systems, phase transition, and coagulation. Nonlocal operators have also been used in image processing.[27,36] In addition, we witness a strong interest for PD, its applications, and related nonlocal problems addressing, for instance, conditioning analysis, domain decomposition, and variational theory,[5–7] discretization,[1,7,25,62] numerical methods,[15,17,50] nonlinear PD,[26,37] nonlinearity in nonlocal wave equations,[19,20] well-posedness in various forms,[5,6,8,18,22–24,39,40,67] and other aspects.[12,28,41,53]

It is part of the folklore in physics that the point particle model, which is the root for *locality* in physics, is the cause of unphysical singular behavior in the description of the underlying phenomena. This fact is a strong indication that, in the long run, the development of nonlocal theories is necessary for description of natural phenomena. Similar set of operator theory tools used for local problems can be transferred to study nonlocal problems because operator theory does not discern the locality or nonlocality of the governing operator. This article adds valuable tools to the arsenal of methods to analyze nonlocal problems.

The equation of interest falls into the class of abstract evolution equations, more precisely, abstract linear wave equations. Methods from operator theory are ideal to treat such equations. One can directly gain access to powerful tools such as functional calculus for bounded self-adjoint operators, spectral theorems (of densely defined, linear, and self-adjoint operators in Hilbert spaces), and strong resolvent convergence of operators. Well-posedness of the initial value problem, conservation of energy, stability of solutions (merely determined by the spectrum) are all immediately available; see Theorem 1 and Corollary 2. Furthermore, representation of solutions can easily be constructed through functional calculus for bounded operators; see our main result 3.

There are many studies[6–8,18,24,39,40,56,67] that aim to establish a connection between the nonlocal operator and the classical one. We denote this nonlocal-to-local connection by $NL2L$. PD formulation utilizes a distance parameter δ, called the horizon, in order to limit interactions to nearby points. The source of nonlocality is the horizon and it is embedded in the support of the micromodulus function. Studies turn to the limit of vanishing nonlocality, i.e., $\delta \to 0$ when they aim to establish $NL2L$. The exact expression of $NL2L$ is lost due to taking limit. We take general micromodulus functions in $L^1(\mathbb{R}^n)$, not necessarily parametrized by δ. Without resorting to a $\delta \to 0$ argument, we identify the exact expression of the $NL2L$ connection, what we call as main result 1.

Identifying the $NL2L$ connection leads to a fruitful research direction because it helped us to extend the construction for unbounded domains to the bounded domain case by using the same function of the classical operator used in PD. That way, we kept a close proximity to PD in the bounded domain case. More importantly, $NL2L$ connection turned out to be the notable result that *the governing operator is a bounded function of the classical (local) operator*.[69] This has far reaching consequences. It enables the incorporation of local boundary conditions into nonlocal theories; see the companion papers.[2,3] It takes a lot of effort to mitigate the boundary effects. See the comprehensive discussion [Ref. 38, Chaps. 4, 5, 7, and 12] and the numerical study.[32] Incorporation of local boundary conditions provides a remedy for boundary effects seen in PD.

The rest of the article is structured as follows. We provide mathematical introduction in Section I. In Section II, we set the operator theory framework to treat the nonlocal wave equation. We prove basic properties of the solutions such as well-posedness of the initial value problem and conservation of energy. We provide a representation of the solutions in terms of bounded functions of the governing operator. We study the stability of solutions. We also consider the class of inhomogeneous wave equations and prove well-posedness of the corresponding initial value problem as well as a representation of the solutions in terms of bounded functions of the governing operator.

In Section III, in the vector-valued case, we note that the governing operator becomes an operator matrix. The generality of operator theory allows a simple extension of the results established for the scalar-valued functions to the vector-valued ones. We prove the boundedness of the entries of the governing operator matrix. The proof is natural due to operator theory again, because it relies on a well-known criterion for integral operators. We present a "diagonalization" of the matrix entries. This is accomplished by employing the unitary Fourier transform and connecting the entries to maximal multiplication operators.

In Section IV, we collect the statements of the three main results:

- **Main Result 1:** Nonlocal operator is a bounded function of the classical operator.
- **Main Result 2:** Strong convergence of nonlocal solutions to classical ones through strong resolvent convergence.
- **Main Result 3:** Representation of the solution in terms of spherical Bessel functions.

In Section V, we construct two examples to demonstrate the usefulness of main result 3. We give explicit representation of the solutions in terms of Bessel functions. Since the explicit representation is available, we easily compute the solutions by symbolic computation, depict the resulting solutions of PD wave equation, and compare to the classical solutions.

In Section VI, we collect the proofs of the three main results together with related supporting material. In particular, in Section VI A, utilizing Fourier transforms, we turn the nonlocal governing operator into maximal multiplication operator. This process can be viewed as a form of "diagonalization." The spectra of maximal multiplication operators are well understood. In addition, the functional calculus associated to maximal multiplication operators is known. Spectra and functional calculus allow the construction of the functional calculi for the governing operator, which in turn is used to prove that the governing operators with spherically symmetric micromoduli are functions of the Laplace operator. In Section VI B, we prove strong convergence of nonlocal solutions to classical ones through the notion of strong resolvent convergence of functions of the classical operator. Different types of resolvent convergences, i.e., norm, strong, and weak, are available; see Ref. 65 for comparison. We utilize the notion of strong resolvent convergence in order to obtain strong convergence of solution operators. This allows us to prove main result 3. We give examples of sequences of micromoduli that are instances of this result. In Section VI C, we consider the calculation of the solution of the wave equation. Since the governing operator is bounded, holomorphic functions of that operator can be represented in the form of power series in the operator. Then, we give a representation of holomorphic functions, present in the solution of the initial value problem, utilizing the fact that the governing operator is a sum of two commuting operators. We establish main result 3 by discovering that the corresponding power series can be given in terms of a series of Bessel functions. In addition, we provide an error estimate for the series representation. We use this estimate in the plots of solutions of the PD wave equation; see Figures 1 and 2. In Section VI D, we give the explicit representation of the solutions of inhomogeneous wave equations and prove well-posedness of the corresponding initial value problem. We conclude in Section VII.

Next, we present the formal system of linear PD wave equations in n-space dimensions [Ref. 55, Eq. (54)], $n \in \mathbb{N}^*$,

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) = \int_{\mathbb{R}^n} C(x'-x)\cdot(u(x',t)-u(x,t))\,dx' + b(x,t).$$

Here, "·" indicates matrix multiplication, or equivalently by the system

$$\rho \frac{\partial^2 u_j}{\partial t^2}(x,t) = \sum_{k=1}^{n}\int_{\mathbb{R}^n} C_{jk}(x'-x)\cdot(u_k(x',t)-u_k(x,t))\,dx' + b_j(x,t), \tag{1.1}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $C : \mathbb{R}^n \to M(n \times n, \mathbb{R})$ is the micromodulus tensor, assumed to be even and assuming values inside the subspace of symmetric matrices, $\rho > 0$ is the mass density, $b : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the prescribed body force density, and $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the displacement field.

For comparison, e.g., the corresponding wave equation in classical elasticity in 1-space dimension is given by

$$\rho \frac{\partial^2 u}{\partial t^2} = E\frac{\partial^2 u}{\partial x^2} + b, \tag{1.2}$$

where $E > 0$ is the so called "Young's modulus," and describing compression waves in a rod.

If $j,k \in \{1,\ldots,n\}$ and $C_{jk} \in L^1(\mathbb{R}^n)$, we can rewrite (1.1) as

$$\rho \frac{\partial^2 u_j}{\partial t^2}(x,t) = -\sum_{k=1}^{n}\left\{\left[\int_{\mathbb{R}^n} C_{jk}(x')dx'\right] u_k(x,t) - (C_{jk} * u_k(\cdot,t))(x)\right\} + b_j(x,t), \tag{1.3}$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $j \in \{1,\ldots,n\}$ where $*$ denotes the convolution product. The system (1.3) is the starting point for a functional analytic interpretation, which leads on a well-posed initial value problem. For this purpose, we use methods from operator theory; see, e.g., Refs. 11 and 47.

II. OPERATOR-THEORETIC TREATMENT OF SYSTEMS OF WAVE EQUATIONS

Analogous to the majority of evolution equations from classical and quantum physics, (1.3) can be treated with methods from operator theory, see, e.g., Refs. 11 and 47 for substantiation of this claim and Ref. 29 for applications of operator theory in engineering. More specifically, this system

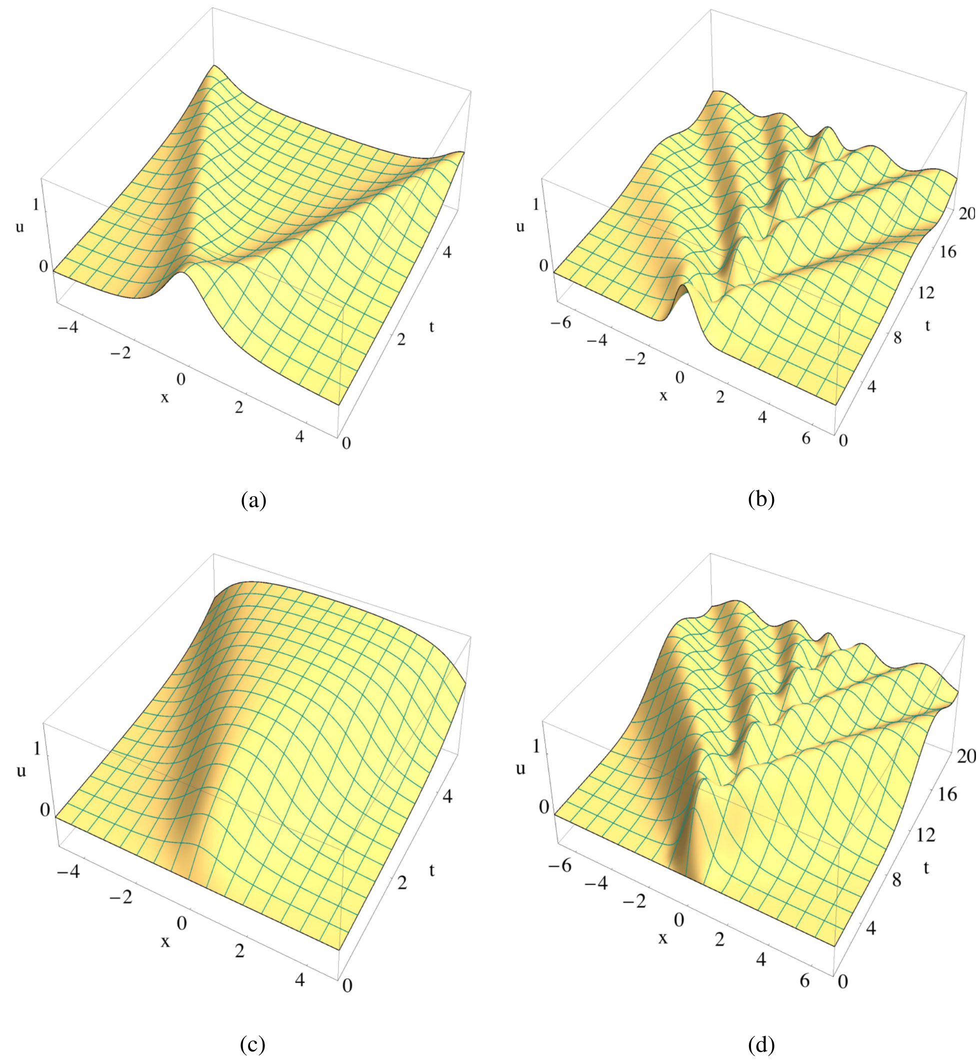

FIG. 1. Evolution of the local and nonlocal wave equation solutions with vanishing initial velocity ((a) and (b)) and vanishing initial displacement ((c) and (d)). For (a) and (c), we use $\rho = E = 1$, $b = 0$, values in (1.2). For (b) and (d), we use $c = a = 1$, $\rho = 1$, $\sigma = 1$, $\sigma_d = 1/2$ values in Example 10. Solutions are generated using symbolic computation. The infinite series in (5.1) is truncated after adding 46 terms. No difference has been observed visually if more terms are added.

falls into the class of abstract linear wave equations from Theorem 1. For the proof of this theorem see, e.g., Ref. 11 [Theorem 2.2.1 and Corollary 2.2.2]. Special cases of this theorem are proved in Refs. 30 and 42 and Ref. 48 [Vol. II]. Statements and proofs make use of the spectral theorems of (densely defined, linear and) self-adjoint operators in Hilbert spaces, including the concept of functions of such operators, see, e.g., Ref. 48 [Vol. I], or standard books on functional analysis, such as Refs. 49 and 66. These methods are also used throughout the paper.

This section provides the basic properties of the solutions of abstract wave equations of the form (2.1). Theorem 1 gives the well-posedness of the initial value problem for a class of abstract wave equations, conservation of energy, and a representation of the solutions in terms of bounded functions of the governing operator. Theorem 3 provides special solutions of the associated class of inhomogeneous wave equations. Together with Theorem 1, these solutions provide the well-posedness of the initial value problem of the latter equations as well as a representation of the solutions in terms of bounded functions of the governing operator.

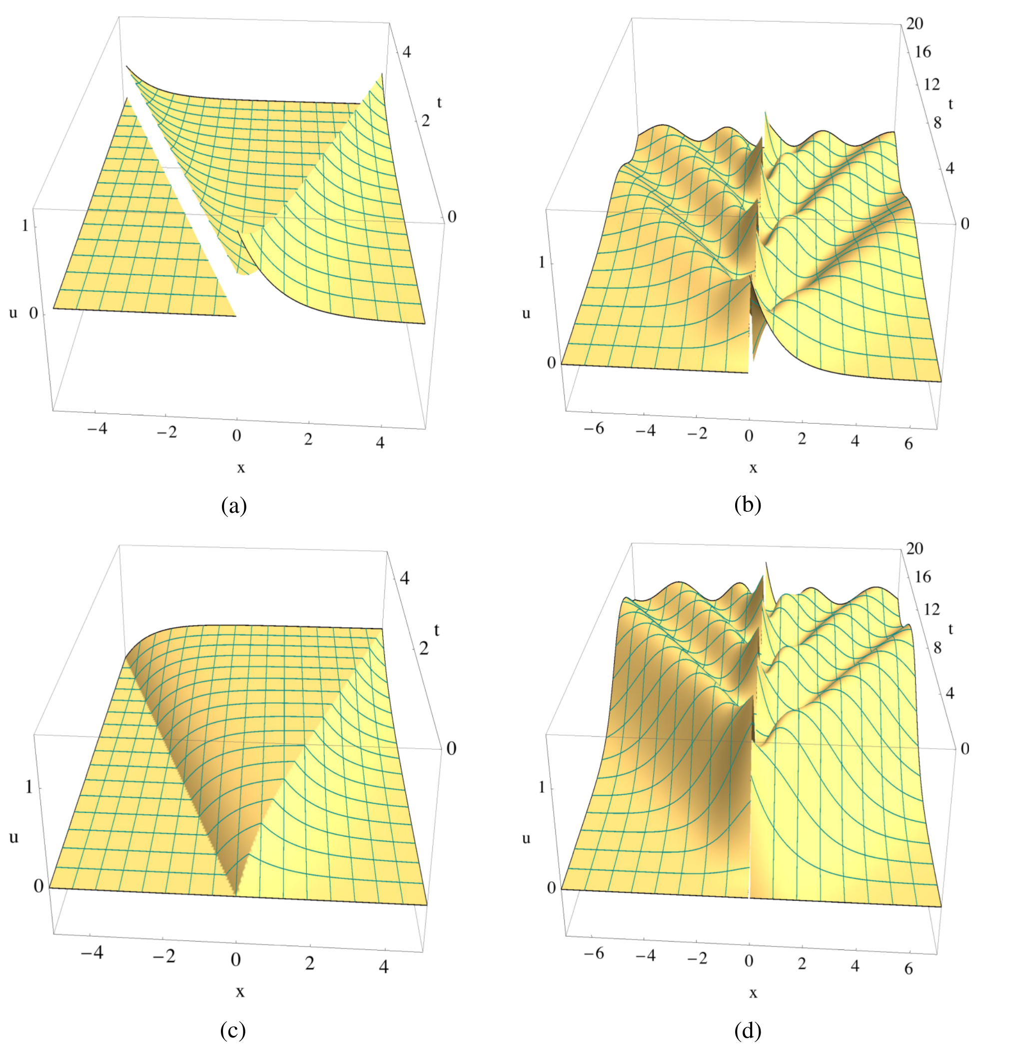

FIG. 2. Evolution of the local and nonlocal wave equation solutions with discontinuous initial displacement ((a) and (b)) and discontinuous initial velocity ((c) and (d)). For (a) and (c), we use $\rho = E = 1$, $b = 0$, values in (1.2). For (b) and (d), we use $c = a = 1$, $\rho = 1$, $\sigma = 1$, and $b = \epsilon = 1$ values in Example 11. Solutions are generated using symbolic computation. The infinite series in (5.2) is truncated after adding 46 terms. No difference has been observed visually if more terms are added.

Theorem 1 (Wave Equations). *Let $(X, \langle|\rangle)$ be some non-trivial complex Hilbert space. Furthermore, let $A : D(A) \to X$ be some densely defined, linear, semibounded self-adjoint operator in X with spectrum $\sigma(A)$. Finally, let $\xi, \eta \in D(A)$.*

(i) *Then there is a unique twice continuously differentiable map $u : \mathbb{R} \to X$ assuming values in $D(A)$ and satisfying*

$$u''(t) = -A\,u(t) \tag{2.1}$$

for all $t \in \mathbb{R}$ as well as

$$u(0) = \xi,\ u'(0) = \eta.$$

(ii) *For this u, the corresponding energy function $E_u : \mathbb{R} \to \mathbb{R}$, defined by*

$$E_u(t) := \frac{1}{2}\left(\langle u'(t)|u'(t)\rangle + \langle u(t)|Au(t)\rangle\right)$$

for all $t \in \mathbb{R}$, is constant.

(iii) *Moreover, this u is represented by the following solution operators for all $t \in \mathbb{R}$:*[70]

$$u(t) = \cos\left(t\sqrt{A}\right)\,\xi + \frac{\sin\left(t\sqrt{A}\right)}{\sqrt{A}}\eta. \tag{2.2}$$

Proof. See the proofs of Ref. 11 [Theorem 2.2.1 and Corollary 2.2.2]. □

Moreover, if A is positive, the solutions of (2.1) are stable, i.e., there are no solutions that are growing exponentially in the norm.

Corollary 2 (Stability of solutions). If A is positive, then for every $t \in \mathbb{R}$, we have

$$\|u(t)\| \leqslant \|\xi\| + |t| \cdot \|\eta\|.$$

Proof. The proof is obvious. □

Duhamel's principle leads to a solution of (2.1) for vanishing data, the proof of the well-posedness and a representation of the solutions of the initial value problem of the inhomogeneous equation,

$$u''(t) = -A\,u(t) + b(t), \quad t \in \mathbb{R}.$$

For simplicity, the corresponding subsequent Theorem 3 assumes that A is in addition positive, which is the most relevant case for applications because otherwise there are exponentially growing solutions, indicating that the system is unstable. The same statement is true if $\sigma(A)$ is only bounded from below. On the other hand, Theorem 3 can also be obtained by application of the corresponding well-known more general theorem for strongly continuous semigroups; see, e.g., Ref. 11 [Theorem 4.6.2]. We give a direct proof of Theorem 3 in Section VI D, which does not rely on methods from the theory of strongly continuous semigroups. For the definition of weak integration; see, e.g., Ref. 11 [Sec. 3.2].

Theorem 3 (Solutions of inhomogeneous wave equations). *Let $(X,\langle|\rangle)$, $A : D(A) \to X$, $\sigma(A)$ be as in Theorem 1 and, in addition, A be positive. Let $b : \mathbb{R} \to X$ be a continuous map, assuming values in $D(A^2)$ such that Ab, $A^2 b$ are continuous. Then, $v : \mathbb{R} \to X$, for every $t \in \mathbb{R}$ defined by*

$$v(t) := \int_{I_t} \frac{\sin\left((t-\tau)\sqrt{A}\right)}{\sqrt{A}}\, b(\tau)\, d\tau,$$

where $\int$ denotes weak integration in X,

$$I_t := \begin{cases} [0,t] & \text{if } t \geqslant 0 \\ [t,0] & \text{if } t < 0 \end{cases},$$

is twice continuously differentiable, assumes values in $D(A)$, is such that

$$v''(t) + A\,v(t) = b(t), \quad t \in \mathbb{R},$$
$$v(0) = v'(0) = 0.$$

Proof. See Section VI D. □

III. THE GOVERNING OPERATOR AND PROPERTIES

The standard data space for the classical wave equation is a L^2-space with constant weight, on a non-empty open subset of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, for instance, $L^2_{\mathbb{C}}(\mathbb{R})$ for a bar of infinite extension in 1-space dimension. It turns out that the classical data spaces are suitable also as data spaces for peridynamics, for instance, again $L^2_{\mathbb{C}}(\mathbb{R})$ for a bar of infinite extension in 1-space dimension, composed of a "linear peridynamic material." This simplifies the discussion of the convergence of peridynamic solutions to classical solutions.

In the following, we represent (1.3) in the form of (2.1), where the governing operator A is an "operator matrix," consisting of sums of multiples of the identity and convolution operators, as indicated in (1.3). These matrix entries will turn out to be pairwise commuting. The following remark provides some known relevant information on operator matrices of bounded operators. On the other hand, we avoid explicit matrix notation.

Remark 4 (Operator matrices). If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(X, \langle|\rangle)$ *a non-trivial* $\mathbb{K}$*-Hilbert space,* $(A_{jk})_{j,k\in\{1,...,n\}}$ *a family of elements of* $L(X, X)$.

(i) *Then by*

$$A(\xi_1,\dots,\xi_n) := \left(\sum_{k=1}^{n} A_{1k}\xi_k, \dots, \sum_{k=1}^{n} A_{nk}\xi_k\right)$$

for every $(\xi_1,\dots,\xi_n) \in X^n$*, there is defined a bounded linear operator with adjoint* A^* *given by*

$$A^*(\xi_1,\dots,\xi_n) = \left(\sum_{k=1}^{n} A^*_{k1}\xi_k, \dots, \sum_{k=1}^{n} A^*_{kn}\xi_k\right)$$

for every $(\xi_1,\dots,\xi_n) \in X^n$.

(ii) *If the members of* $(A_{jk})_{j,k\in\{1,...,n\}}$ *are pairwise commuting, then A is bijective if and only if* $\det(A)$ *is bijective, where*

$$\det(A) := \sum_{\sigma\in S_n} sign(\sigma)\, A_{1\sigma(1)} \cdots A_{n\sigma(n)},$$

S_n *denotes the set of permutations of* $\{1,\dots,n\}$,

$$sign(\sigma) := \prod_{i,j=1, i<j}^{n} sign(\sigma(j) - \sigma(i))$$

for all $\sigma \in S_n$ *and sign denotes the signum function.*

The basic properties of the entries of the operator matrix are given in the following lemma. In fact, these operators turn out to be bounded linear operators on $L^2_{\mathbb{C}}(\mathbb{R}^n)$. Hence, the boundedness and self-adjointness of A follows from those of A_C. The boundedness of A has appeared in various forms[5,6,8,18,23,24,67] and sometimes for special class of kernel functions. We give a result using kernel functions that are in $L^1(\mathbb{R}^n)$ by utilizing a well-known criterion for integral operators; see, e.g., corollary to Ref. 64 [Theorem 6.24].

Lemma 5 (Boundedness of matrix entries). Let $n \in \mathbb{N}^*$, $\rho > 0$ *and* $C \in L^1(\mathbb{R}^n)$ *be even. Then,*

$$A_C f := \frac{1}{\rho}\left[\left(\int_{\mathbb{R}^n} C\, dv^n\right).f - C * f\right], \tag{3.1}$$

for every $f \in L^2_{\mathbb{C}}(\mathbb{R}^n)$*, where* $*$ *denotes the convolution product, there is defined a self-adjoint bounded linear operator on* $L^2_{\mathbb{C}}(\mathbb{R}^n)$ *with operator norm* $\|A_C\|$ *satisfying*

$$\|A_C\| \leqslant \frac{1}{\rho}\left(\left|\int_{\mathbb{R}} C\, dv^n\right| + \|C\|_1\right) \leqslant \frac{2\|C\|_1}{\rho}.$$

Proof. By a consequence of a well-known criterion for integral operators on L^2-spaces, see, e.g., Corollary to Ref. 64 [Theorem 6.24]. □

For other governing operators related to A_C on bounded domains, see Ref. 4.

IV. MAIN RESULTS

The following are the three main results in this paper:

- **Main Result 1:** Nonlocal operator is a bounded function of the classical operator.
- **Main Result 2:** Strong convergence of nonlocal solutions to classical ones through strong resolvent convergence.
- **Main Result 3:** Representation of the solution in terms of spherical Bessel functions.

Our first main result establishes the connection between the nonlocal operator and the classical one. We identify the exact expression of this connection. Namely, the governing operator A_C in (3.1) of the peridynamic wave equation is a bounded function of the classical governing operator in (1.2). This observation enables the comparison of peridynamic solutions to those of classical elasticity. In addition, it enables the generalization of peridynamic-type operators from $\mathbb{R}^n$ to bounded domains as functions of the corresponding classical operator. This is the subject of our companion papers.[2,3]

Theorem 6 (Main result 1). *Let $n \in \mathbb{N}^*$, $\mathcal{L}_n$ be the closure of the positive symmetric, essentially self-adjoint operator in $L^2_{\mathbb{C}}(\mathbb{R}^n)$, given by*

$$\left(C_0^\infty(\mathbb{R}^n, \mathbb{C}) \to L^2_{\mathbb{C}}(\mathbb{R}^n),\ f \mapsto -\frac{E}{\rho} \triangle f\right),$$

where $\rho > 0$ and $E > 0$. In addition, for $n > 1$, let C be spherically symmetric, i.e., such that

$$C \circ R = C,$$

for every $R \in SO(n)$, where $SO(n)$ denotes the map of group of special orthogonal transformations on $\mathbb{R}^n$. Then

$$A_C = \left\{\frac{1}{\rho}\,[(F_1 C)(0) - F_1 C] \circ \iota\right\}(\mathcal{L}_n),$$

where $\iota : [0, \infty) \to \mathbb{R}^n$ is defined by

$$\iota(s) := \left(\sqrt{\frac{\rho}{E}\, s}\right).e_1,$$

for every $s \geqslant 0$ and $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$.

Employing the notion of strong resolvent convergence, our second main result gives the strong convergence of solutions of the governing equation to that of the classical equation. There are results that establish the convergence of the peridynamic operator to the classical operator with vanishing nonlocality; see Refs. 6, 7, 24, and 56. More important for applications is the corresponding convergence of solutions. This question has also been pointed out in Ref. 24 [p. 862] for the convergence of PD solutions to that of Navier equation. The studies[8,18,67] considered convergence of solutions. In particular, Refs. 39 and 40 put an effort to establish strong convergence of solutions. One can find a survey of existing convergence results in Ref. 21 [Chap. 4]. The tool that we develop for this purpose is the notion of strong resolvent convergence[64,65] used in Theorem 7. To the best of our knowledge, our study provides the first exploitation of strong resolvent convergence of the functions of the classical operator to obtain strong convergence of solutions.

Theorem 7 (Main result 2). *Let $(X, \langle | \rangle)$ be a non-trivial complex Hilbert space and $A : D(A) \to X$ a densely defined, linear, and self-adjoint operator with spectrum $\sigma(A)$. Furthermore, let $f_1, f_2, \dots$ be a sequence of real-valued functions in $U^s_{\mathbb{C}}(\sigma(A))$[71] that is everywhere on $\sigma(A)$ pointwise convergent to $id_{\sigma(A)}$, and for which there is $M > 0$ such that*

$$|f_\nu| \leqslant M[(1 + |\ |)|_{\sigma(A)}]$$

for all $\nu \in \mathbb{R}$. Then for every $g \in BC(\mathbb{R}, \mathbb{C})$, where $BC(\mathbb{R}^n, \mathbb{C})$ is the space of complex-valued bounded continuous functions on $\mathbb{R}^n$,

$$s - \lim_{\nu \to \infty} [g|_{\sigma(f_\nu(A))}](f_\nu(A)) = [g|_{\sigma(A)}](A),$$

where for every $\nu \in \mathbb{N}^*$, $\sigma(f_\nu(A))$ denotes the spectrum of $f_\nu(A)$.

Remark 8. The mechanism to obtain strong convergence of nonlocal solutions to classical ones is as follows. We first choose a sequence of micromoduli C_ν so that

$$f_{C_\nu}(\lambda) := F_1(C_\nu)(0) - F_1(C_\nu)(\lambda) \tag{4.1}$$

is pointwise convergent to $\mathrm{id}_{\sigma(\mathcal{L}_1)}$ and that satisfies condition (6.1), where $\mathcal{L}_1$ is the classical governing operator in 1 dimension defined in Theorem 6. We choose $A = \mathcal{L}_1$ and f_{C_ν} in (4.1). Then, a well-known result [Ref. 64, Theorem 9.16] indicates that Lemma 15 implies strong resolvent convergence of $f_{C_\nu}(\mathcal{L}_1)$ to $\mathcal{L}_1$. Then, Theorem 7 implies that for every $g \in BC(\mathbb{R},\mathbb{C})$, we obtain the following strong convergence:

$$s - \lim_{\nu\to\infty} [g|_{\sigma(f_{C_\nu}(\mathcal{L}_1))}](f_{C_\nu}(\mathcal{L}_1)) = [g|_{\sigma(\mathcal{L}_1)}](\mathcal{L}_1). \tag{4.2}$$

We denote the nonlocal operator corresponding to micromoduli C_ν by $A_{C_\nu} := f_{C_\nu}(\mathcal{L}_1)$. Then, we apply (4.2) to solution operators g in (2.2), namely, $\cos(t\sqrt{A_{C_\nu}})$ and $\sin(t\sqrt{A_{C_\nu}})/\sqrt{A_{C_\nu}}$, both of which are in $BC(\mathbb{R},\mathbb{C})$. Consequently, we obtain strong convergence of nonlocal solution operators $[g|_{\sigma(f_{C_\nu}(\mathcal{L}_1))}](f_{C_\nu}(\mathcal{L}_1))$ in (3.1) to the classical solution operators $[g|_{\sigma(\mathcal{L}_1)}](\mathcal{L}_1)$.

As an application, we establish strong convergence of solutions generated by standard examples of micromoduli C_ν in the literature, given in (6.3) and (6.4); see Lemmata 16 and 18.

Our third main result is the discovery that the solution can be represented in terms of a series of Bessel functions. This property can be exploited in obtaining numerical representation of the solutions. We utilized this for the solutions depicted in Section V.

Theorem 9 (Main result 3). *Let $n \in \mathbb{N}^*$, $\rho > 0$, $C \in L^1(\mathbb{R}^n)$ be even such that*

$$c := \int_{\mathbb{R}^n} C \, dv^n > 0.$$

Then, for A_C in (3.1) and $t \in \mathbb{R}$, the solutions have the following representation:

$$\begin{aligned} \cos\left(t\sqrt{A_C}\right) f &= \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2/\rho}\,)^{k+1} j_{k-1}(\sqrt{ct^2/\rho}\,)\, c^{-k}.C^k * f, \\ \frac{\sin\left(t\sqrt{A_C}\right)}{\sqrt{A_C}} f &= t \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2/\rho}\,)^{k} j_{k}(\sqrt{ct^2/\rho}\,)\, c^{-k}.C^k * f, \end{aligned} \tag{4.3}$$

for every $f \in L^2_{\mathbb{C}}(\mathbb{R}^n)$, where the spherical Bessel functions $j_0, j_1, \ldots$ are defined as in Ref. 43,

$$j_{-1}(z) := \frac{\cos(z)}{z}, \quad z \in \mathbb{C}^*,$$

and the members of the sums are defined for $t = 0$ by continuous extension.

V. EXAMPLES AND ILLUSTRATIONS OF MAIN RESULT 3

As an application of main result 3, we construct examples of the nonlocal wave equation to illustrate the separation of waves and propagation of discontinuities, Examples 10 and 11, respectively. The depicted solutions in Figures 1 and 2 were computed symbolically using spherical Bessel function representation of the solutions. The infinite series in (5.1) and (5.2), respectively, are truncated after adding 46 terms. Visually, we do not observe a difference in the solutions if more terms are added.

For constructing the case that characterizes separation of waves, we choose an example that involves a micromodulus and input function both of which are normal distributions with mean value zero and standard deviation σ and σ_d, respectively. Similar micromodulus functions have also been used in Refs. 41, 63, and 68.

Example 10. For $\rho, \sigma, \sigma_d, a > 0$, we define $C_\sigma \in L^1(\mathbb{R})$ and $f \in L^2_{\mathbb{C}}(\mathbb{R})$ by

$$C_\sigma := \frac{a}{\sqrt{2\pi}\,\sigma}\, e^{-[1/(2\sigma^2)].\mathrm{id}^2_{\mathbb{R}}}, \quad f := \frac{1}{\sqrt{2\pi}\sigma_d}\, e^{-[1/(2\sigma_d^2)].\mathrm{id}^2_{\mathbb{R}}}.$$

Then for $k \in \mathbb{N}^*$

$$
\begin{aligned}
&F_1 C_\sigma = a e^{-(\sigma^2/2).\mathrm{id}_{\mathbb{R}}^2}, \ F_1 C_\sigma^k = (F_1 C_\sigma)^k = a^k e^{-k(\sigma^2/2).\mathrm{id}_{\mathbb{R}}^2}, \\
&F_2(C_\sigma^k * f) = (F_1 C_\sigma^k) \cdot F_2 f = \frac{a^k}{\sqrt{2\pi}} e^{-k(\sigma^2/2).\mathrm{id}_{\mathbb{R}}^2} \cdot e^{-(\sigma_d^2/2).\mathrm{id}_{\mathbb{R}}^2} \\
&= \frac{a^k}{\sqrt{2\pi}} e^{-[(k\sigma^2+\sigma_d^2)/2].\mathrm{id}_{\mathbb{R}}^2} = \frac{1}{\sqrt{2\pi}} F_1 \frac{a^k}{\sqrt{2\pi}\sqrt{k\sigma^2+\sigma_d^2}} e^{-\{1/[2(k\sigma^2+\sigma_d^2)]\}.\mathrm{id}_{\mathbb{R}}^2} \\
&= F_2 \frac{a^k}{\sqrt{2\pi}\sqrt{k\sigma^2+\sigma_d^2}} e^{-\{1/[2(k\sigma^2+\sigma_d^2)]\}.\mathrm{id}_{\mathbb{R}}^2}
\end{aligned}
$$

and hence

$$
C_\sigma^k * f = \frac{a^k}{\sqrt{2\pi}\sqrt{k\sigma^2+\sigma_d^2}} e^{-\{1/[2(k\sigma^2+\sigma_d^2)]\}.\mathrm{id}_{\mathbb{R}}^2}.
$$

Since

$$
c = \int_{\mathbb{R}} C_\sigma \, dv^1 = a > 0,
$$

we conclude from Theorem 9 that for A_C in (3.1) and $t \in \mathbb{R}$,

$$
\begin{aligned}
\cos\left(t\sqrt{A_C}\right) f &= \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{at^2/\rho})^{k+1} j_{k-1}(\sqrt{at^2/\rho}) \frac{1}{\sqrt{2\pi}\sqrt{k\sigma^2+\sigma_d^2}} e^{-\{1/[2(k\sigma^2+\sigma_d^2)]\}.\mathrm{id}_{\mathbb{R}}^2}, \\
\frac{\sin\left(t\sqrt{A_C}\right)}{\sqrt{A_C}} f &= t \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{at^2/\rho})^{k} j_{k}(\sqrt{at^2/\rho}) \frac{1}{\sqrt{2\pi}\sqrt{k\sigma^2+\sigma_d^2}} e^{-\{1/[2(k\sigma^2+\sigma_d^2)]\}.\mathrm{id}_{\mathbb{R}}^2}.
\end{aligned} \tag{5.1}
$$

We depict and compare the solutions of the classical and nonlocal wave equations in Figures 1 and 2. In the classical case, as expected, we observe the propagation of waves along characteristics; see Figures 1(a) and 1(c) for vanishing initial velocity and displacement, respectively. In the nonlocal case, we observe repeated separation of waves and an oscillation at the center of the initial pulse; see Figures 1(b) and 1(d) for vanishing initial velocity and displacement, respectively. Solution wave patterns were also reported earlier[23,25] for the same micromodulus function in Example 11.

Now, we study the propagation of discontinuity in the data for classical and nonlocal wave equations in the following example.

Example 11. As in the previous example, for $\rho, \sigma, a, b, \varepsilon > 0$, we define $C_\sigma \in L^1(\mathbb{R})$ by

$$
C_\sigma := \frac{a}{\sqrt{2\pi}\,\sigma} e^{-[1/(2\sigma^2)].\mathrm{id}_{\mathbb{R}}^2}.
$$

Then

$$
F_1 C_\sigma = a e^{-(\sigma^2/2).\mathrm{id}_{\mathbb{R}}^2},
$$

and for $k \in \mathbb{N}^*$,

$$
F_1 C_\sigma^k = (F_1 C_\sigma)^k = a^k e^{-k(\sigma^2/2).\mathrm{id}_{\mathbb{R}}^2} = F_1 \frac{a^k}{\sqrt{2\pi k}\,\sigma} e^{-[1/(2k\sigma^2)].\mathrm{id}_{\mathbb{R}}^2},
$$

and hence

$$
C_\sigma^k = \frac{a^k}{\sqrt{2\pi}\sqrt{k\sigma^2}} e^{-[1/(2k\sigma^2)].\mathrm{id}_{\mathbb{R}}^2}.
$$

Furthermore, we define $f \in L^2_{\mathbb{C}}(\mathbb{R})$ by

$$f := b\, e^{-\varepsilon.\mathrm{id}_{\mathbb{R}}} \cdot \chi_{[0,\infty)}.$$

Then for $x \in \mathbb{R}$,

$$\begin{aligned}(C^k_\sigma * f)(x) &= \frac{a^k b}{\sqrt{2\pi}\sqrt{k\sigma^2}} \int_0^\infty e^{-[(x-y)^2/(2k\sigma^2)]} \cdot e^{-\varepsilon y}\, dy \\ &= \frac{2a^k b}{\pi}\, e^{\left(\frac{\varepsilon\sigma}{2}\sqrt{2k}\right)^2} e^{-\varepsilon x}\, \mathrm{erfc}\left(\frac{\varepsilon\sigma}{2}\sqrt{2k} - \frac{x}{\sigma\sqrt{2k}}\right),\end{aligned}$$

where erfc *denotes the error function defined according to DLMF.*[43] *We note for $x \in \mathbb{R}$ that*

$$\lim_{k\to 0} \frac{a^k b}{2}\, e^{\left(\frac{\varepsilon\sigma}{2}\sqrt{2k}\right)^2} e^{-\varepsilon x}\, \mathrm{erfc}\left(\frac{\varepsilon\sigma}{2}\sqrt{2k} - \frac{x}{\sigma\sqrt{2k}}\right) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{b}{2} & \text{if } x = 0 \\ b e^{-\varepsilon x} & \text{if } x > 0 \end{cases}.$$

Since

$$c = \int_{\mathbb{R}} C_\sigma\, dv^1 = a > 0,$$

we conclude from Theorem 9 that for $t \in \mathbb{R}$,

$$\begin{aligned}\cos\left(t\sqrt{A_C}\right) f &= \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{at^2/\rho})^{k+1} j_{k-1}(\sqrt{at^2/\rho})\, f_k, \\ \frac{\sin\left(t\sqrt{A_C}\right)}{\sqrt{A_C}} f &= t \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{at^2/\rho})^{k} j_{k}(\sqrt{at^2/\rho})\, f_k,\end{aligned} \tag{5.2}$$

for A_C in (3.1), where for every $x \in \mathbb{R}$ and $k \in \mathbb{N}^*$,

$$\begin{aligned} f_0 &:= b\, e^{-\varepsilon.\mathrm{id}_{\mathbb{R}}} \cdot \chi_{[0,\infty)}, \\ f_k(x) &:= \frac{2b}{\pi}\, e^{\left(\frac{\varepsilon\sigma}{2}\sqrt{2k}\right)^2} e^{-\varepsilon x}\, \mathrm{erfc}\left(\frac{\varepsilon\sigma}{2}\sqrt{2k} - \frac{x}{\sigma\sqrt{2k}}\right) \\ &= \frac{b}{\sqrt{2\pi}\sqrt{k\sigma^2}}\, e^{-\varepsilon x} \int_{-\infty}^{x} e^{-u^2/(2k\sigma^2)} \cdot e^{\varepsilon u}\, du.\end{aligned}$$

In the classical wave equation, as expected, discontinuities propagate along the characteristics; see Figures 2(a) and 2(c) for vanishing initial velocity and displacement, respectively. On the other hand, in the nonlocal case, the discontinuity remains in the same place for all time; see Figures 2(b) and 2(d) for vanishing initial velocity and displacement, respectively. This confirms the results given in Ref. 63.

VI. PROOFS AND RELATED RESULTS

A. Proof of main result 1 (nonlocal operator is a bounded function of the classical operator)

For the study of the spectral properties of the matrix entries, needed for the application of the results from Section II, we use Fourier transformations. This step parallels the common procedure for constant coefficient differential operators on $\mathbb{R}^n, n \in \mathbb{N}^*$. With the help of the unitary Fourier transform F_2, Theorem 13 represents the matrix entries as maximal multiplication operators. This process can be viewed as a form of "diagonalization" of the entries. Also, since bounded maximal multiplication operators commute, the entries commute pairwise. The spectra of maximal multiplication operators are well understood, leading to Corollary 14. Also, the functional calculus which is associated to maximal multiplication operators is known and allows the construction of the functional calculi of the entries. The latter is used in the proof of Theorem 6 which proves that matrix entries corresponding to spherically symmetric micromoduli are functions of the Laplace operator.

Assumption 12. In the following, for $n \in \mathbb{N}^*$, F_2 denotes the unitary Fourier transformation on $L^2_{\mathbb{C}}(\mathbb{R}^n)$ which, for every rapidly decreasing test function $f \in \mathscr{S}_{\mathbb{C}}(\mathbb{R})$, is defined by

$$(F_2 f)(k) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ik\cdot id_{\mathbb{R}^n}} f\, dv^n, \quad k \in \mathbb{R}^n.$$

Also, we denote by F_1 the map from $L^1_{\mathbb{C}}(\mathbb{R}^n)$ to $C_\infty(\mathbb{R}^n, \mathbb{C})$, the space of continuous functions on $\mathbb{R}^n$ vanishing at infinity, which for every $f \in L^1_{\mathbb{C}}(\mathbb{R}^n)$, is defined by

$$(F_1 f)(k) := \int_{\mathbb{R}^n} e^{-ik\cdot id_{\mathbb{R}^n}} f\, dv^n, \quad k \in \mathbb{R}^n.$$

Theorem 13 (Fourier transforms of the entries). *Let*

$$T_{\frac{1}{\rho}[(F_1C)(0) - F_1C]}$$

denote the maximal multiplication operator by the bounded continuous function

$$\frac{1}{\rho}[(F_1C)(0) - F_1C]$$

on $L^2_{\mathbb{C}}(\mathbb{R}^n)$. Then

$$F_2 \circ A_C \circ F_2^{-1} = T_{\frac{1}{\rho}[(F_1C)(0) - F_1C]}.$$

Proof. First, we define an operator $K := C \circ (p_1 - p_2)$, with projections $p_1, p_2 : \mathbb{R}^{2n} \to \mathbb{R}^n$, where

$$p_1(x_1, \ldots, x_n, y_1, \ldots, y_n) := (x_1, \ldots, x_n),\ p_2(x_1, \ldots, x_n, y_1, \ldots, y_n) := (y_1, \ldots, y_n),$$

for $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$. Since C is even, K is symmetric. Furthermore, for every $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$,

$$K(x, \cdot) = C(x - \cdot) = C(\cdot - x),\ K(\cdot, y) = C(\cdot - y) \in L^1(\mathbb{R}^n).$$

K induces a self-adjoint bounded linear integral operator $\text{Int}(K)$ on $L^2_{\mathbb{C}}(\mathbb{R}^n)$ defined by

$$[\text{Int}(K) f](x) := \int_{\mathbb{R}^n} K(x, \cdot) \cdot f\, dv^n = \int_{\mathbb{R}^n} C(x - \cdot) \cdot f\, dv^n = (C * f)(x).$$

We note that for every $L^1_{\mathbb{C}}(\mathbb{R}^n) \cap L^2_{\mathbb{C}}(\mathbb{R}^n)$,

$$\begin{aligned}[F_2 \circ \text{Int}(K)] f = F_2(C * f) &= \frac{1}{(2\pi)^{n/2}}.F_1(C * f) = \frac{1}{(2\pi)^{n/2}}.(F_1 C)(F_1 f) \\ &= (F_1 C)(F_2 f) = [T_{F_1 C} \circ F_2] f.\end{aligned}$$

Since $L^1_{\mathbb{C}}(\mathbb{R}^n) \cap L^2_{\mathbb{C}}(\mathbb{R}^n)$ is dense in $L^2_{\mathbb{C}}(\mathbb{R}^n)$, the bounded linear operators $F_2 \circ \text{Int}(K)$ and $T_{F_1C} \circ F_2$ coincide on a dense subspace of $L^2_{\mathbb{C}}(\mathbb{R}^n)$, they coincide on the whole of $L^2_{\mathbb{C}}(\mathbb{R}^n)$. Consequently, the result follows from

$$[F_2 \circ \text{Int}(K)] f = [T_{F_1C} \circ F_2] f, \quad f \in L^2_{\mathbb{C}}(\mathbb{R}^n).$$

□

We finally arrive at the proof of main result 1.

Proof. First, we note that

$$F_2 \circ \mathcal{L}_n \circ F_2^{-1} = T_{\frac{E}{\rho}|\ |^2},$$

where $T_{\frac{E}{\rho}|\ |^2}$ denotes the maximal multiplication operator in $L^2_{\mathbb{C}}(\mathbb{R}^n)$ by the function $\frac{E}{\rho}|\ |^2$. In particular, this implies that the spectrum of $\mathcal{L}_n$, $\sigma(\mathcal{L}_n)$, is given by $[0, \infty)$ and for every $g \in U^s_{\mathbb{C}}([0, \infty))$ that

$$g(\mathcal{L}_n) = F_2^{-1} \circ T_{g \circ \left(\frac{E}{\rho}|\ |^2\right)} \circ F_2,$$

where $T_{g\circ\left(\frac{E}{\rho}|\,|^2\right)}$ denotes the maximal multiplication operator on $L^2_{\mathbb{C}}(\mathbb{R}^n)$ by the function

$$g \circ \left(\frac{E}{\rho}|\,|^2\right).$$

Furthermore, we note that $(F_1C)(0) - F_1C \in BC(\mathbb{R}^n,\mathbb{R})$ and that $(F_1C)(0) - F_1C$ is even, since for every $k \in \mathbb{R}^n$

$$\begin{aligned}
(F_1C)(-k) &= \int_{\mathbb{R}^n} e^{ik\cdot \mathrm{id}_{\mathbb{R}^n}} C\, dv^n = \int_{\mathbb{R}^n} e^{-ik\cdot \mathrm{id}_{\mathbb{R}^n}} [C \circ (-\mathrm{id}_{\mathbb{R}^n})]\, dv^n \\
&= \int_{\mathbb{R}^n} e^{-ik\cdot \mathrm{id}_{\mathbb{R}^n}} C\, dv^n = (F_1C)(k), \\
(F_1C)(k) &= \frac{1}{2}\left[\int_{\mathbb{R}^n} e^{-ik\cdot \mathrm{id}_{\mathbb{R}^n}} C\, dv^n + \int_{\mathbb{R}^n} e^{ik\cdot \mathrm{id}_{\mathbb{R}^n}} C\, dv^n\right] \\
&= \int_{\mathbb{R}^n} \cos(k\cdot \mathrm{id}_{\mathbb{R}^n})\, C\, dv^n, \\
(F_1C)(0) - (F_1C)(k) &= \int_{\mathbb{R}^n} [1 - \cos(k\cdot \mathrm{id}_{\mathbb{R}^n})]\, C\, dv^n = 2\int_{\mathbb{R}^n} \sin^2\left(\frac{k}{2}\cdot \mathrm{id}_{\mathbb{R}^n}\right) C\, dv^n.
\end{aligned}$$

Furthermore for $n > 1$, we note that

$$\begin{aligned}
(F_1C)(R(k)) &= \int_{\mathbb{R}^n} e^{-iR(k)\cdot \mathrm{id}_{\mathbb{R}^n}} C\, dv^n = \int_{\mathbb{R}^n} e^{-iR(k)\cdot R}\, (C \circ R)\, dv^n \\
&= \int_{\mathbb{R}^n} e^{-ik\cdot \mathrm{id}_{\mathbb{R}^n}}\, (C \circ R)\, dv^n = \int_{\mathbb{R}^n} e^{-ik\cdot \mathrm{id}_{\mathbb{R}^n}} C\, dv^n = (F_1C)(k)
\end{aligned}$$

for every $R \in SO(n)$ and $k \in \mathbb{R}^n$ and hence that

$$(F_1C)(k) = (F_1C)(|k|.e_1)$$

for every $k \in \mathbb{R}^n$. In particular,

$$\frac{1}{\rho}\,[(F_1C)(0) - F_1C] \circ \iota \in U^s_{\mathbb{R}}([0,\infty))$$

and

$$\begin{aligned}
&\left\{\frac{1}{\rho}\,[(F_1C)(0) - F_1C] \circ \iota\right\}(\mathcal{L}_n) = F_2^{-1} \circ T_{\left\{\frac{1}{\rho}[(F_1C)(0)-F_1C]\circ\iota\right\}\circ\left(\frac{E}{\rho}|\,|^2\right)} \circ F_2 \\
&= F_2^{-1} \circ T_{\frac{1}{\rho}[(F_1C)(0)-F_1C]\circ(|\,|.e_1)} \circ F_2 = F_2^{-1} \circ T_{\frac{1}{\rho}[(F_1C)(0)-F_1C]} \circ F_2 = A_C.
\end{aligned}$$

□

We give the spectrum and point spectrum of A_C. The spectrum is an essential ingredient in constructing functional calculus for A_C; see Lemma 20.

Corollary 14 (Spectral Properties of A_C).

$$\sigma(A_C) = \overline{\mathrm{Ran}\frac{1}{\rho}.[(F_1C)(0) - F_1C]},$$

$$\sigma_p(A_C) = \left\{\lambda \in \mathbb{R} : \left\{k \in \mathbb{R} : \frac{1}{\rho}.[(F_1C)(0) - (F_1C)(k)] = \lambda\right\} \textit{ is no Lebesgue null set}\right\},$$

where the overline denotes the closure in $\mathbb{R}$. Finally, for every $\lambda \in \sigma(A_C)$, $A_C - \lambda$ is not surjective.

Proof. The result is a consequence of well-known properties of multiplication operators. □

B. Proof of main result 2 (strong convergence of nonlocal solutions to classical ones through strong resolvent convergence)

Lemma 15 gives conditions for the convergence of bounded functions of a self-adjoint operator to converge to that operator, which implies strong resolvent convergence and also the strong convergence of the same bounded continuous function of each member of the sequence against that bounded continuous function of the self-adjoint operator [Ref. 64, Theorem 9.16]; see Theorem 7.

Lemma 15 (Convergence of bounded functions of a self-adjoint operator to that operator). Let $(X, \langle|\rangle)$ *be a non-trivial complex Hilbert space and* $A : D(A) \to X$ *a densely defined, linear, and self-adjoint operator with spectrum* $\sigma(A)$. *Furthermore, let* $f_1, f_2, \dots$ *be a sequence in* $U^s_{\mathbb{C}}(\sigma(A))$ *that is everywhere on* $\sigma(A)$ *pointwise convergent to* $\mathrm{id}_{\sigma(A)}$, *and for which there is* $M > 0$ *such that*

$$|f_\nu| \leqslant M[(1 + |\ |)|_{\sigma(A)}] \tag{6.1}$$

for all $\nu \in \mathbb{R}$. *Then*

$$\lim_{\nu\to\infty} f_\nu(A)\xi = A\xi, \quad \xi \in D(A). \tag{6.2}$$

Proof. Let $\xi \in D(A)$ and ψ_ξ the corresponding spectral measure. According to the spectral theorem for densely defined, self-adjoint linear operators in Hilbert spaces, $\mathrm{id}^2_{\mathbb{R}}$ is ψ_ξ-summable and

$$\begin{aligned}
&\|f_\mu(A)\xi - f_\nu(A)\xi\|^2 = \|(f_\mu - f_\nu)(A)\xi\|^2 \\
&= \langle (f_\mu - f_\nu)(A)\xi | (f_\mu - f_\nu)(A)\xi\rangle = \langle \xi | |f_\mu - f_\nu|^2(A)\xi\rangle \\
&= \int_{\sigma(A)} |f_\mu - f_\nu|^2\, d\psi_\xi = \|f_\mu - f_\nu\|^2_{2,\psi_\xi} = \|f_\mu - \mathrm{id}_{\sigma(A)} + \mathrm{id}_{\sigma(A)} - f_\nu\|^2_{2,\psi_\xi} \\
&\leqslant \left(\|f_\mu - \mathrm{id}_{\sigma(A)}\|_{2,\psi_\xi} + \|\mathrm{id}_{\sigma(A)} - f_\nu\|_{2,\psi_\xi}\right)^2,
\end{aligned}$$

for $\mu, \nu \in \mathbb{N}^*$. As a consequence of the pointwise convergence of $f_1, f_2, \dots$ on $\sigma(A)$ to $\mathrm{id}_{\sigma(A)}$, (6.1) and Lebesgue's dominated convergence theorem, it follows that

$$\lim_{\mu\to\infty} \|f_\mu - \mathrm{id}_{\sigma(A)}\|_{2,\psi_\xi} = 0$$

and hence that $f_1(A)\xi, f_2(A)\xi, \dots$ is a Cauchy sequence in X. Since $(X, \|\ \|)$ is in particular complete, the latter implies that $f_1(A)\xi, f_2(A)\xi, \dots$ is convergent in $(X, \|\ \|)$. Furthermore,

$$\begin{aligned}
&\langle \xi | \lim_{\nu\to\infty} f_\nu(A)\xi\rangle = \lim_{\nu\to\infty} \langle \xi | f_\nu(A)\xi\rangle = \lim_{\nu\to\infty} \int_{\sigma(A)} f_\nu\, d\psi_\xi \\
&= \int_{\sigma(A)} \mathrm{id}_{\sigma(A)}\, d\psi_\xi = \langle \xi | A\xi\rangle,
\end{aligned}$$

where again the pointwise convergence of $f_1, f_2, \dots$ on $\sigma(A)$ to $\mathrm{id}_{\sigma(A)}$, (6.1), Lebesgue's dominated convergence theorem and the spectral theorem for densely defined, self-adjoint linear operators in Hilbert spaces has been applied. From the polarization identity for $\langle|\rangle$, it follows that

$$\langle \xi | \lim_{\nu\to\infty} f_\nu(A)\eta\rangle = \langle \xi | A\eta\rangle$$

for all $\xi, \eta \in D(A)$. Since $D(A)$ is dense in X, the latter implies that

$$\langle \xi | \lim_{\nu\to\infty} f_\nu(A)\eta\rangle = \langle \xi | A\eta\rangle$$

for all $\xi \in X$, $\eta \in D(A)$ and hence for every $\eta \in D(A)$ that

$$\lim_{\nu\to\infty} f_\nu(A)\eta = A\eta.$$

□

Now, main result 2, Theorem 7, is a consequence of Lemma 15, and, for example, Ref. 48 [Vol. I, Theorems 8.20 and 8.25].

We apply our main result 2 to standard examples treated in the literature. Hence, we easily obtain strong resolvent convergence for these cases, i.e., Lemmas 16 and 18. They provide sequences of micromoduli which satisfy the conditions of Lemma 15. Lemma 16 has also been treated in Refs. 41 and 61 and Lemma 18 has been treated in Ref. 41. Remark 8 applies Theorem 7 to the sequences of micromoduli from Lemmas 16 and 18. As a consequence, for fixed data and $t \in \mathbb{R}$, the solutions of the initial value problem at time t corresponding to the members of each sequence of micromoduli converge in $L^2_{\mathbb{C}}(\mathbb{R})$ to the corresponding classical solution at time t.

Lemma 16. For every $\nu \in \mathbb{N}^*$, we define $C_\nu \in L^1(\mathbb{R})$ by

$$C_\nu := 3E\nu^3 \chi_{[-\frac{1}{\nu},\frac{1}{\nu}]}. \tag{6.3}$$

Then, the result (6.2) preparing for strong convergence of solutions is satisfied

$$\lim_{\nu\to\infty} \left\{ \frac{1}{\rho} [(F_1 C_\nu)(0) - F_1 C] \circ \iota \right\} (\mathcal{L}_1) f = \mathcal{L}_1 f, \quad f \in D(\mathcal{L}_1) = W^2_{\mathbb{C}}(\mathbb{R}).$$

Remark 17. By applying Theorem 7 as in Remark 8, we obtain that the solutions of the nonlocal operator A_{C_ν} strongly converge to that of the local operator for the micromoduli C_ν in (6.3).

Proof. For $\nu \in \mathbb{N}^*$,

$$F_1 C_\nu = 6E\nu^3 \frac{\sin(\nu^{-1}.\mathrm{id}_{\mathbb{R}})}{\mathrm{id}_{\mathbb{R}}}.$$

Furthermore, for $\nu \in \mathbb{N}^*$, $\lambda \geqslant 0$,

$$\frac{1}{\rho}[(F_1 C_\nu)(0) - F_1 C] \circ \iota(\lambda) = \frac{1}{\rho}[(F_1 C_\nu)(0) - F_1 C_\nu] \left(\sqrt{\frac{\rho}{E}\lambda} \right) = \frac{6E\nu^2}{\rho} \left[1 - \frac{\sin(\nu^{-1}.\mathrm{id}_{\mathbb{R}})}{\nu^{-1}.\mathrm{id}_{\mathbb{R}}} \right] \left(\sqrt{\frac{\rho}{E}\lambda} \right)$$

and $k > 0$

$$1 - \frac{\sin(k/\nu)}{k/\nu} = \nu \int_0^{1/\nu} [1 - \cos(kx)]\, dx = \int_0^1 [1 - \cos(ku/\nu)]\, du = \int_0^1 \left[\int_0^{k/\nu} u \sin(uy)\, dy \right] du = \frac{k}{\nu} \int_0^1 \left[\int_0^1 u \sin(kuv/\nu)\, dv \right] du = \frac{k^2}{\nu^2} \int_{[0,1]^2} u^2 v \frac{\sin(kuv/\nu)}{kuv/\nu}\, dudv$$

and hence that

$$\nu^2 \left[1 - \frac{\sin(k/\nu)}{k/\nu} \right] = k^2 \int_{[0,1]^2} u^2 v \frac{\sin(kuv/\nu)}{kuv/\nu}\, dudv.$$

From the latter, we conclude with the help of Lebesgue's dominated convergence theorem that

$$\lim_{\nu\to\infty} \nu^2 \left[1 - \frac{\sin(k/\nu)}{k/\nu} \right] = \frac{k^2}{6}$$

as well as that

$$\left| \nu^2 \left[1 - \frac{\sin(k/\nu)}{k/\nu} \right] \right| \leqslant k^2 \int_{[0,1]^2} u^2 v \left| \frac{\sin(kuv/\nu)}{kuv/\nu} \right| dudv \leqslant k^2 \int_{[0,1]^2} u^2 v\, dudv = \frac{k^2}{6}.$$

In particular, we conclude for $\lambda \geqslant 0$ that

$$\lim_{\nu\to\infty} \frac{1}{\rho}[(F_1 C_\nu)(0) - F_1 C] \circ \iota(\lambda) = \frac{6E\nu^2}{\rho} \cdot \frac{\rho\lambda}{6E\nu^2} = \lambda$$

as well as that

$$\left| \frac{1}{\rho}[(F_1 C_\nu)(0) - F_1 C] \circ \iota(\lambda) \right| \leqslant \frac{6E}{\rho} \cdot \frac{1}{6} \frac{\rho\lambda}{E} = \lambda.$$

Finally, we conclude the result from Lemma 15. □

Lemma 18. For every $\nu \in \mathbb{N}^*$, we define $C_\nu \in L^1(\mathbb{R})$ by

$$C_\nu := \frac{2E\nu^3}{\sqrt{2\pi}} e^{-(\nu^2/2).\mathrm{id}^2_{\mathbb{R}}} = 2E\nu^2 \cdot \frac{\nu}{\sqrt{2\pi}} e^{-(\nu^2/2).\mathrm{id}^2_{\mathbb{R}}}. \tag{6.4}$$

Then, the result (6.2) preparing for strong convergence of solutions is satisfied,

$$\lim_{\nu\to\infty}\left\{\frac{1}{\rho}\,[(F_1C_\nu)(0)-F_1C]\circ\iota\right\}(\mathcal{L}_1)f=\mathcal{L}_1 f,\quad f\in D(\mathcal{L}_1)=W^2_{\mathbb{C}}(\mathbb{R}).$$

Remark 19. By applying Theorem 7 as in Remark 8, we obtain that the solutions of the nonlocal operator A_{C_ν} strongly converge to that of the local operator for the micromoduli C_ν in (6.4).

Proof. For $\nu\in\mathbb{N}^*$, $\lambda\geqslant 0$,

$$\begin{aligned}
&F_1C_\nu = 2E\nu^2\cdot e^{-[1/(2\nu^2)].\mathrm{id}^2_{\mathbb{R}}},\\
&\frac{1}{\rho}\,[(F_1C_\nu)(0)-F_1C_\nu]\circ\iota(\lambda)=\frac{1}{\rho}\,[(F_1C_\nu)(0)-F_1C_\nu]\left(\sqrt{\frac{\rho}{E}\,\lambda}\right)\\
&=\frac{2E\nu^2}{\rho}\left\{1-e^{-[1/(2\nu^2)].\mathrm{id}^2_{\mathbb{R}}}\right\}\left(\sqrt{\frac{\rho}{E}\,\lambda}\right)
\end{aligned}$$

and $k\geqslant 0$

$$\nu^2[1-e^{-k^2/(2\nu^2)}]=\nu^2\int_0^{k^2/(2\nu^2)}e^{-u}\,du=\int_0^{k^2/2}e^{-v/\nu^2}\,dv.$$

From the latter, we conclude for $k\geqslant 0$, with the help of Lebesgue's dominated convergence theorem, that

$$\lim_{\nu\to\infty}\nu^2[1-e^{-k^2/(2\nu^2)}]=\frac{k^2}{2}$$

as well as that

$$\left|\nu^2[1-e^{-k^2/(2\nu^2)}]\right|\leqslant\frac{k^2}{2}$$

and hence for $\lambda\geqslant 0$ that

$$\begin{aligned}
&\lim_{\nu\to\infty}\frac{1}{\rho}\,[(F_1C_\nu)(0)-F_1C_\nu]\circ\iota(\lambda)=\frac{2E}{\rho}\frac{\rho}{2E}\,\lambda=\lambda,\\
&\left|\frac{1}{\rho}\,[(F_1C_\nu)(0)-F_1C_\nu]\circ\iota(\lambda)\right|\leqslant\frac{2E}{\rho}\,\frac{\rho}{2E}\,\lambda=\lambda.
\end{aligned}$$

Finally, we conclude the result from Lemma 15. □

C. Proof of main result 3 (representation of the solution in terms of spherical Bessel functions)

In this section, the goal is to prove that the solutions of the nonlocal wave equation given in (2.2) can be expressed in terms of spherical Bessel functions. This result has a practical implication because it allows symbolic computation of solutions. Indeed, using symbolic computation, we generate solutions of the nonlocal wave equation and illustrate how separation of waves and propagation of discontinuities occur; see Figures 1 and 2. In addition, one can use the explicit representations of solutions for benchmarking numerical results which helps in the development of numerical methods for computing approximate solutions. Such benchmarking could be used, for instance, for verifying and validating numerical solutions.

Main result 1 indicates that the governing peridynamic operator is bounded. Hence, functions of that operator in (2.2) can be represented in the form of power series in the governing operator due to functional calculus of self-adjoint and bounded operators. In Lemma 20, we prove that holomorphic functional calculus is available for power series representation. In Lemma 21, we apply this representation to the functions present in the solution of the initial value problem of the homogeneous wave equation. Lemmas 20 and 21 can be viewed as straightforward applications of the spectral theorems for densely defined, self-adjoint linear operators in Hilbert spaces.

Lemma 20 (Holomorphic functional calculus). Let $(X, \langle|\rangle)$ *be a non-trivial complex Hilbert space,* $A \in L(X,X)$ *self-adjoint, and* $\sigma(A) \subset \mathbb{R}$ *the (non-empty, compact) spectrum of* A*. Furthermore, let* $R > \|A\|$ *and* $f : U_R(0) \to \mathbb{C}$ *be holomorphic. Then, the sequence*

$$\left(\frac{f^{(k)}(0)}{k!}.A^k\right)_{k\in\mathbb{N}}$$

is absolutely summable in $L(X,X)$ *and*

$$(f|_{\sigma(A)})(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}.A^k.$$

Proof. First, we note that according to Taylor's theorem, general properties of power series, and the compactness of $\sigma(A)$ that

$$\left(\frac{f^{(k)}(0)}{k!}.z^k\right)_{k\in\mathbb{N}}$$

is absolutely summable for every $z \in U_R(0)$ as well as, since $\sigma(A) \subset B_{\|A\|}(0) \subset U_R(0)$, that the sequence of continuous functions

$$\left(\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}.(\mathrm{id}_{\mathbb{R}}|_{\sigma(A)})^n\right)_{n\in\mathbb{N}}$$

converges uniformly to the continuous function $f|_{\sigma(A)}$. In particular, since $\|A\| < R$, this implies that the sequence

$$\left(\frac{f^{(k)}(0)}{k!}.A^k\right)_{k\in\mathbb{N}}$$

is absolutely summable in $L(X,X)$, and it follows from the spectral theorem for bounded self-adjoint operators in Hilbert spaces that

$$\left(\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}.(\mathrm{id}_{\mathbb{R}}|_{\sigma(A)})^n\right)(A) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}.A^k,$$

as well as that

$$(f|_{\sigma(A)})(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}.A^k.$$

□

Now, we give the exact representation of the terms present in (2.2) in terms of power series in A.

Lemma 21 (Power series representation). Let $(X, \langle|\rangle)$ *be a non-trivial complex Hilbert space,* $\sqrt{\ }$ *the complex square-root function, with domain* $\mathbb{C} \setminus ((-\infty, 0] \times \{0\})$*.* $A \in L(X,X)$ *self-adjoint and* $\sigma(A) \subset \mathbb{R}$ *the (non-empty, compact) spectrum of* A*. For every* $t \in \mathbb{R}$*, the sequences*

$$\left((-1)^k \frac{t^{2k}}{(2k)!}.A^k\right)_{k\in\mathbb{N}}, \quad \left((-1)^k \frac{t^{2k+1}}{(2k+1)!}.A^k\right)_{k\in\mathbb{N}}$$

are absolutely summable in $L(X,X)$ *and*

$$\cos\left(t\sqrt{A}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} A^k,$$

$$\frac{\sin\left(t\sqrt{A}\right)}{\sqrt{A}} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} A^k.$$

Proof. We note that for every $t \in \mathbb{R}$,

$$\cos(t\sqrt{}\,) : \mathbb{C} \setminus ((-\infty, 0] \times \{0\}) \to \mathbb{C},$$
$$\cosh(t\sqrt{} \circ (-\mathrm{id}_{\mathbb{C}})\,) : \mathbb{C} \setminus ([0, \infty) \times \{0\}) \to \mathbb{C}$$

are holomorphic functions such that

$$\cos(t\sqrt{z}\,) = \sum_{k=0}^{\infty} (-1)^k \frac{(t\sqrt{z})^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} z^k,$$
$$\cosh(t\sqrt{-z}\,) = \sum_{k=0}^{\infty} \frac{(t\sqrt{-z})^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} z^k$$

for every $z \in \mathbb{C} \setminus ((-\infty, 0] \times \{0\})$ and $z \in \mathbb{C} \setminus [0, \infty) \times \{0\})$, respectively. Furthermore,

$$\frac{\sin(t\sqrt{}\,)}{\sqrt{}} : \mathbb{C} \setminus ((-\infty, 0] \times \{0\}) \to \mathbb{C},$$
$$\frac{\sinh(t\sqrt{}\,)}{\sqrt{}} \circ (-\mathrm{id}_{\mathbb{C}}) : \mathbb{C} \setminus ([0, \infty) \times \{0\}) \to \mathbb{C}$$

are holomorphic functions such that

$$\frac{\sin(t\sqrt{z}\,)}{\sqrt{z}} = \frac{1}{\sqrt{z}} \sum_{k=0}^{\infty} (-1)^k \frac{(t\sqrt{z})^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} z^k,$$
$$\frac{\sinh(t\sqrt{-z}\,)}{\sqrt{-z}} = \frac{1}{\sqrt{-z}} \sum_{k=0}^{\infty} \frac{(t\sqrt{-z})^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} z^k$$

for every $z \in \mathbb{C} \setminus ((-\infty, 0] \times \{0\})$ and $z \in \mathbb{C} \setminus [0, \infty) \times \{0\})$, respectively. Then, it follows from Lemma 20 that the sequences

$$\left((-1)^k \frac{t^{2k}}{(2k)!} .A^k\right)_{k \in \mathbb{N}}, \quad \left((-1)^k \frac{t^{2k+1}}{(2k+1)!} .A^k\right)_{k \in \mathbb{N}}$$

are absolutely summable in $L(X, X)$ and that

$$\cos\left(t\sqrt{A}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} A^k,$$
$$\frac{\sin\left(t\sqrt{A}\right)}{\sqrt{A}} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} A^k.$$

□

The goal here was to show that (2.2) can be expressed by spherical Bessel functions. In order to arrive at a Bessel function representation, we first connect (2.2) to generalized hypergeometric functions. In Theorem 22, we give a general result which involves two arbitrary bounded commuting operators A and B. In Theorem 24, we will apply this result to special commuting operators, multiple of the identity operator, and a convolution, i.e., $c - C$, because the governing operator is of this form. In addition, we expect that the expansion below can be used for large time asymptotic of the solutions of the nonlocal wave equation, a research direction beyond the scope of this paper.

Theorem 22. *Let $(X, \langle|\rangle)$ be a non-trivial complex Hilbert space, $\sqrt{}$ the complex square-root function, with domain $\mathbb{C} \setminus ((-\infty, 0] \times \{0\})$. $A, B \in L(X, X)$ self-adjoint such that $[A, B] = 0$ and $\sigma(A), \sigma(A + B) \subset \mathbb{R}$ the (non-empty, compact) spectra of A and $A + B$, respectively. Then*

$$\cos\left(t\sqrt{A+B}\right) = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{t^{2k}}{(2k)!} \cdot \left\{\left[{}_0F_1\left(-; k + \frac{1}{2}; -\frac{t^2}{4}.\mathrm{id}_{\sigma(A)}\right)\right](A)\right\} B^k,$$
$$\frac{\sin\left(t\sqrt{A+B}\right)}{\sqrt{A+B}} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{t^{2k+1}}{(2k+1)!} \cdot \left\{\left[{}_0F_1\left(-; k + \frac{3}{2}; -\frac{t^2}{4}.\mathrm{id}_{\sigma(A)}\right)\right](A)\right\} B^k,$$

where ${}_0F_1$ denotes the generalized hypergeometric function, defined as in Ref. 43.

Proof. In a first step, we note for every $t \in \mathbb{R}$ that the family

$$\left((-1)^{k+l}\binom{k+l}{l}\frac{t^{2(k+l)}}{[2(k+l)]!}A^kB^l\right)_{(k,l)\in\mathbb{N}^2}$$

is absolutely summable in $L(X,X)$, since for $(k,l)\in\mathbb{N}^2$

$$\begin{aligned}&\left\|(-1)^{k+l}\binom{k+l}{l}\frac{t^{2(k+l)}}{[2(k+l)]!}A^kB^l\right\| \leqslant \frac{t^{2(k+l)}}{[2(k+l)]!}\binom{k+l}{l}\|A\|^k\|B\|^l\\ &= \frac{(k+l)!}{l!k![2(k+l)]!}\big(t^2\|A\|\big)^k\big(t^2\|B\|\big)^l \leqslant \frac{1}{k!}\big(t^2\|A\|\big)^k\frac{1}{l!}\big(t^2\|B\|\big)^l\end{aligned}$$

and hence for every finite subset $S \subset \mathbb{N}^2$,

$$\sum_{(k,l)\in S}\left\|(-1)^{k+l}\binom{k+l}{l}\frac{t^{2(k+l)}}{[2(k+l)]!}A^kB^l\right\| \leqslant \exp\left(t^2\|A\|\right)\exp\left(t^2\|B\|\right).$$

Also, we note that the family

$$\left((-1)^{k+l}\binom{k+l}{l}\frac{t^{2(k+l)+1}}{[2(k+l)+1]!}A^kB^l\right)_{(k,l)\in\mathbb{N}^2}$$

is absolutely summable in $L(X,X)$, since for $(k,l)\in\mathbb{N}^2$,

$$\begin{aligned}&\left\|(-1)^{k+l}\binom{k+l}{l}\frac{t^{2(k+l)+1}}{[2(k+l)+1]!}A^kB^l\right\|_2 \leqslant \frac{|t|^{2(k+l)+1}}{[2(k+l)+1]!}\binom{k+l}{l}\|A\|^k\|B\|^l\\ &= |t|\,\frac{(k+l)!}{l!k![2(k+l)+1]!}\big(t^2\|A\|\big)^k\big(t^2\|B\|\big)^l \leqslant |t|\,\frac{1}{k!}\big(t^2\|A\|\big)^k\frac{1}{l!}\big(t^2\|B\|\big)^l,\end{aligned}$$

leading to

$$\sum_{(k,l)\in S}\left\|(-1)^{k+l}\binom{k+l}{l}\frac{t^{2(k+l)+1}}{[2(k+l)+1]!}A^kB^l\right\| \leqslant |t|\exp\left(t^2\|A\|\right)\exp\left(t^2\|B\|\right),$$

for every finite subset $S \subset \mathbb{N}^2$. Hence, we conclude the following:

$$\begin{aligned}&\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}(A+B)^k = \sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^k\frac{t^{2k}}{(2k)!}\binom{k}{l}A^{k-l}B^l\\ &= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}\binom{k}{l}A^{k-l}B^l = \sum_{l=0}^{\infty}\left[\sum_{k=l}^{\infty}(-1)^k\binom{k}{l}\frac{t^{2k}}{(2k)!}A^{k-l}\right]B^l\\ &= \sum_{l=0}^{\infty}\left[\sum_{k=0}^{\infty}(-1)^{k+l}\binom{k+l}{l}\frac{t^{2(k+l)}}{[2(k+l)]!}A^k\right]B^l\\ &= \sum_{l=0}^{\infty}(-1)^lt^{2l}\left[\sum_{k=0}^{\infty}(-1)^k\binom{k+l}{l}\frac{t^{2k}}{[2(k+l)]!}A^k\right]B^l\\ &= \sum_{l=0}^{\infty}(-1)^l\,t^{2l}\left[\sum_{k=0}^{\infty}\frac{(k+l)!}{[2(k+l)]!\cdot l!}\cdot\frac{1}{k!}(-t^2A)^k\right]B^l.\end{aligned}$$

In the following, we show the auxiliary result that for every $k,l\in\mathbb{N}$,

$$\frac{(k+l)!}{[2(k+l)]!\cdot l!} = 2^{-k}\cdot\frac{1}{\prod_{m=0}^{k-1}[2(l+m)+1]}\cdot\frac{1}{(2l)!}. \tag{6.5}$$

The proof proceeds by induction on k. First, we note that

$$\frac{l!}{(2l)!\cdot l!} = 2^{-0}\cdot\frac{1}{\prod_{m=0}^{-1}[2(l+m)+1]}\cdot\frac{1}{(2l)!}.$$

In the following, we assume that (6.5) is true for some $k \in \mathbb{N}$. Then

$$\frac{(k+l+1)!}{[2(k+l+1)]! \cdot l!} = \frac{1}{2} \cdot \frac{1}{2(k+l)+1} \cdot \frac{(k+l)!}{[2(k+l)]! \cdot l!}$$
$$= \frac{1}{2} \cdot \frac{1}{2(k+l)+1} \cdot 2^{-k} \cdot \frac{1}{\prod_{m=0}^{k-1}[2(l+m)+1]} \cdot \frac{1}{(2l)!}$$
$$= 2^{-(k+1)} \cdot \frac{1}{\prod_{m=0}^{k}[2(l+m)+1]} \cdot \frac{1}{(2l)!},$$

and hence (6.5) is true for $k+1$. The equality (6.5) implies for every $k, l \in \mathbb{N}$ that

$$\frac{(k+l)!}{[2(k+l)]! \cdot l!} = 2^{-k} \cdot \frac{1}{\prod_{m=0}^{k-1}[2(l+m)+1]} \cdot \frac{1}{(2l)!}$$
$$= 4^{-k} \cdot \frac{1}{\prod_{m=0}^{k-1}\left(l+m+\frac{1}{2}\right)} \cdot \frac{1}{(2l)!} = 4^{-k} \cdot \frac{1}{(2l)!\left(l+\frac{1}{2}\right)_k}.$$

Hence

$$\sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}}{(2k)!}(A+B)^k = \sum_{l=0}^{\infty}(-1)^l t^{2l}\left[\sum_{k=0}^{\infty}\frac{1}{(2l)!\left(l+\frac{1}{2}\right)_k} \cdot \frac{1}{k!}\left(-\frac{t^2}{4}.A\right)^k\right]B^l$$
$$\sum_{l=0}^{\infty}(-1)^l \frac{t^{2l}}{(2l)!}\left[\sum_{k=0}^{\infty}\frac{1}{\left(l+\frac{1}{2}\right)_k \cdot k!} \cdot \left(-\frac{t^2}{4}.A\right)^k\right]B^l.$$

By definition of the generalized hypergeometric function ${}_0F_1$, for every $l \in \mathbb{N}$, $z \in \mathbb{C}$,

$${}_0F_1\left(-; l+\frac{1}{2}; z\right) = \sum_{k=0}^{\infty}\frac{z^k}{(l+\frac{1}{2})_k \cdot k!}.$$

Hence

$$\sum_{k=0}^{\infty}(-1)^k \frac{t^{2k}}{(2k)!}(A+B)^k = \sum_{l=0}^{\infty}(-1)^l \cdot \frac{t^{2l}}{(2l)!} \cdot \left\{\left[{}_0F_1\left(-; l+\frac{1}{2}; -\frac{t^2}{4}.\mathrm{id}_{\sigma(A)}\right)\right](A)\right\}B^l.$$

Furthermore,

$$\sum_{k=0}^{\infty}(-1)^k \frac{t^{2k+1}}{(2k+1)!}(A+B)^k = \sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^k \frac{t^{2k+1}}{(2k+1)!}\binom{k}{l}A^{k-l}B^l$$
$$= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}(-1)^k \frac{t^{2k+1}}{(2k+1)!}\binom{k}{l}A^{k-l}B^l = \sum_{l=0}^{\infty}\left[\sum_{k=l}^{\infty}(-1)^k\binom{k}{l}\frac{t^{2k+1}}{(2k+1)!}A^{k-l}\right]B^l$$
$$= \sum_{l=0}^{\infty}\left[\sum_{k=0}^{\infty}(-1)^{k+l}\binom{k+l}{l}\frac{t^{2(k+l)+1}}{[2(k+l)+1]!}A^k\right]B^l$$
$$= \sum_{l=0}^{\infty}(-1)^l t^{2l+1}\left[\sum_{k=0}^{\infty}(-1)^k\binom{k+l}{l}\frac{t^{2k}}{[2(k+l)+1]!}A^k\right]B^l$$
$$= t\sum_{l=0}^{\infty}(-1)^l \cdot t^{2l}\left[\sum_{k=0}^{\infty}\frac{(k+l)!}{[2(k+l)+1]! \cdot l!} \cdot \frac{1}{k!}.(-t^2A)^k\right]B^l.$$

Since for every $k, l \in \mathbb{N}$,

$$\frac{(k+l)!}{[2(k+l)+1]! \cdot l!} = \frac{1}{2} \cdot \frac{1}{k+l+\frac{1}{2}} \cdot \frac{(k+l)!}{[2(k+l)]! \cdot l!}$$
$$= \frac{1}{2} \cdot \frac{1}{k+l+\frac{1}{2}} \cdot 4^{-k} \cdot \frac{1}{(2l)!(l+\frac{1}{2})_k} = \frac{1}{2} \cdot 4^{-k} \cdot \frac{1}{(2l)!\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right)_k}$$
$$= 4^{-k} \cdot \frac{1}{(2l+1)!\left(l+\frac{3}{2}\right)_k},$$

we conclude that

$$\begin{aligned}
&\sum_{k=0}^{\infty}(-1)^k \, \frac{t^{2k+1}}{(2k+1)!} \, (A+B)^k \\
&= t \sum_{l=0}^{\infty}(-1)^l \cdot t^{2l} \left[\sum_{k=0}^{\infty} \frac{1}{(2l+1)! \, \left(l+\frac{3}{2}\right)_k} \cdot \frac{1}{k!} . \left(-\frac{t^2}{4}.A\right)^k \right] B^l \\
&= \sum_{l=0}^{\infty}(-1)^l \cdot \frac{t^{2l+1}}{(2l+1)!} \left[\sum_{k=0}^{\infty} \frac{1}{\left(l+\frac{3}{2}\right)_k \cdot k!} . \left(-\frac{t^2}{4}.A\right)^k \right] B^l \\
&= \sum_{l=0}^{\infty}(-1)^l \cdot \frac{t^{2l+1}}{(2l+1)!} . \left\{ \left[{}_0F_1\left(-; l+\frac{3}{2}; -\frac{t^2}{4}.\mathrm{id}_{\sigma(A)}\right)\right](A) \right\} B^l .
\end{aligned}$$

□

We give a connection between generalized hypergeometric and spherical Bessel functions which will lead to main result 3.

Lemma 23. For every $k \in \mathbb{N}$ and $x > 0$,

$$\frac{x^{2k}}{(2k)!} \cdot {}_0F_1\left(-; k+\frac{1}{2}; -\frac{x^2}{4}\right) = \frac{1}{2^k k!} \, x^{k+1} j_{k-1}(x),$$

$$\frac{x^{2k+1}}{(2k+1)!} \cdot {}_0F_1\left(-; k+\frac{3}{2}, -\frac{x^2}{4}\right) = \frac{1}{2^k k!} \, x^{k+1} j_k(|x|),$$

where the spherical Bessel functions $j_0, j_1, \ldots$ are defined as in Ref. 43 and

$$j_{-1}(x) := \frac{\cos(x)}{x}, \quad x > 0.$$

Proof. We note that for every $\nu \in (0,\infty)$, $k \in \mathbb{N}$, and $x > 0$,

$$\begin{aligned}
J_\nu(x) &:= \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, \Gamma(\nu+k+1)} \left(\frac{x^2}{4}\right)^k = \frac{1}{\Gamma(\nu+1)} \cdot \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, \frac{\Gamma(\nu+k+1)}{\Gamma(\nu+1)}} \left(\frac{x^2}{4}\right)^k \\
&= \frac{1}{\Gamma(\nu+1)} \cdot \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, (\nu+1)_k} \left(\frac{x^2}{4}\right)^k = \frac{1}{\Gamma(\nu+1)} \cdot \left(\frac{x}{2}\right)^\nu \cdot {}_0F_1(-; \nu+1, -x^2/4).
\end{aligned}$$

$$\begin{aligned}
j_k(x) &:= \sqrt{\frac{\pi}{2x}} \, J_{k+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} \, \frac{1}{\Gamma(k+\frac{3}{2})} \cdot \left(\frac{x}{2}\right)^{k+\frac{1}{2}} \cdot {}_0F_1(-; k+\frac{3}{2}, -x^2/4) \\
&= \frac{\sqrt{\pi}}{2\Gamma(k+\frac{3}{2})} \cdot \left(\frac{x}{2}\right)^k \cdot {}_0F_1(-; k+\frac{3}{2}, -x^2/4).
\end{aligned}$$

Hence for every $k \in \mathbb{N}$, $x > 0$

$${}_0F_1(-; k+\frac{3}{2}, -x^2/4) = \frac{2\Gamma(k+\frac{3}{2})}{\sqrt{\pi}} \left(\frac{x}{2}\right)^{-k} j_k(x) = 2^{k+1} \left(\frac{1}{2}\right)_{k+1} x^{-k} j_k(x)$$

as well as

$$\begin{aligned}
&\frac{x^{2k+1}}{(2k+1)!} \, {}_0F_1(-; k+\frac{3}{2}, -x^2/4) = \frac{x^{2k+1}}{(2k+1)!} \, 2^{k+1} \left(\frac{1}{2}\right)_{k+1} x^{-k} j_k(x) \\
&= \frac{x^{2k+1}}{(2k+1)!} \, 2^{k+1} \, 2^{-(k+1)} \, \frac{(2k+2)!}{2^{k+1}(k+1)!} x^{-k} j_k(x) = \frac{(2k+2)}{2^{k+1}(k+1)!} \, x^{k+1} j_k(x) \\
&= \frac{1}{2^k k!} \, x^{k+1} j_k(x).
\end{aligned}$$

Furthermore, for every $k \in \mathbb{N}^*$, $x > 0$

$$\frac{x^{2k}}{(2k)!} \cdot {}_0F_1(-;k+\frac{1}{2};-x^2/4) = \frac{x}{2k}\,\frac{1}{2^{k-1}(k-1)!}\,x^k j_{k-1}(x) = \frac{1}{2^k k!}\,x^{k+1} j_{k-1}(x). \tag{6.6}$$

Since for $x > 0$

$$\begin{aligned} {}_0F_1(-;\frac{1}{2},-x^2/4) &= \sum_{k=0}^{\infty} \frac{1}{\left(\frac{1}{2}\right)_k \cdot k!} \cdot \left(-\frac{x^2}{4}\right)^k = \sum_{k=0}^{\infty} \frac{1}{2^{-k} \cdot \frac{(2k)!}{2^k \cdot k!} \cdot k!} \cdot \left(-\frac{x^2}{4}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{4^k}{(2k)!} \cdot \left(-\frac{x^2}{4}\right)^k = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos(x), \end{aligned}$$

the equality (6.6) is true also for $k = 0$, if we define

$$j_{-1}(x) := \frac{\cos(x)}{x}.$$

□

Eventually, we have a representation involving two commuting operators, with one of the operators being a multiple of the identity and a general C which is not necessarily a convolution operator.

Theorem 24. *Let $(X, \langle|\rangle)$ be a non-trivial complex Hilbert space, $\sqrt{\ }$ the complex square-root function, with domain $\mathbb{C} \setminus ((-\infty, 0] \times \{0\})$. $c > 0$, $C \in L(X,X)$ self-adjoint and $\sigma(c - C) \subset \mathbb{R}$ the (non-empty, compact) spectrum of $c - C$. Then for every $t \in \mathbb{R}$,*

$$\cos\left(t\sqrt{c-C}\right) = \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2})^{k+1} j_{k-1}(\sqrt{ct^2}) \left(\frac{1}{c}.C\right)^k,$$

$$\frac{\sin\left(t\sqrt{c-C}\right)}{\sqrt{c-C}} = t \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2})^{k} j_{k}(\sqrt{ct^2}) \left(\frac{1}{c}.C\right)^k,$$

where the spherical Bessel functions $j_0, j_1, \ldots$ are defined as in Ref. 43 and

$$j_{-1}(x) := \frac{\cos(x)}{x}, \quad x > 0$$

and the members of the sums are defined for $t = 0$ by continuous extension.

Proof. Direct consequence of Theorem 22 and Lemma 23. □

Now, main result 3, Theorem 9, is a consequence of Theorem 24. The representations given in Theorems 22, 24, and 9 were used to express the solutions in Examples 10 and 11.

As a side result, we provide an error estimate for the representation in Theorem 24. For the solution plots, Figures 1 and 2, the infinite series is truncated after adding 46 terms, i.e., $N = 45$. The below error estimate has been used to monitor the error. For $N = 45$ and $t = 20$, the error is in the order of 10^{-13}.

Corollary 25 (Error estimates). Let $(X, \langle|\rangle), \sqrt{\ }, c, C, \sigma(c - C), j_{-1}, j_0, j_1, \ldots$ as in Theorem 24 and $N \in \mathbb{N}$. Then for every $t \in \mathbb{R}$,

$$\left\| \cos\left(t\sqrt{c-C}\right) - \sum_{k=0}^{N} \frac{1}{2^k k!} (\sqrt{ct^2})^{k+1} j_{k-1}(\sqrt{ct^2}) \left(\frac{1}{c}.C\right)^k \right\| \leqslant \frac{\pi}{N!} \min\left\{1, \left(\frac{t^2\|C\|}{4}\right)^{N+1}\right\} e^{t^2\|C\|/4},$$

$$\left\| \frac{\sin\left(t\sqrt{c-C}\right)}{\sqrt{c-C}} - t \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2})^{k} j_{k}(\sqrt{ct^2}) \left(\frac{1}{c}.C\right)^k \right\| \leqslant \frac{\pi}{2(N+1)!} |t| \min\left\{1, \left(\frac{t^2\|C\|}{4}\right)^{N+1}\right\} e^{t^2\|C\|/4}.$$

Proof. As a consequence of Theorem 24, for $t \in \mathbb{R}$,

$$\left\| \cos\left(t\sqrt{c-C}\right) - \sum_{k=0}^{N} \frac{1}{2^k k!} (\sqrt{ct^2})^{k+1} j_{k-1}(\sqrt{ct^2}) \left(\frac{1}{c}.C\right)^k \right\|$$
$$\leqslant \sum_{k=N+1}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2})^{k+1} | j_{k-1}(\sqrt{ct^2})| \left(\frac{\|C\|}{c}\right)^k$$
$$\leqslant \sum_{k=N+1}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2})^{k+1} \pi \frac{(\sqrt{ct^2})^{k-1}}{2^k (k-1)!} \left(\frac{\|C\|}{c}\right)^k$$
$$= \pi \sum_{k=N+1}^{\infty} \frac{1}{k!(k-1)!} \left(\frac{t^2\|C\|}{4}\right)^k \leqslant \frac{\pi}{N!} \sum_{k=N+1}^{\infty} \frac{1}{k!} \left(\frac{t^2\|C\|}{4}\right)^k,$$

where the integral representation DLMF 10.54.1 of Ref. 43 (`http://dlmf.nist.gov/10.54`) for spherical Bessel functions has been used. Since

$$\sum_{k=N+1}^{\infty} \frac{1}{k!} \left(\frac{t^2\|C\|}{4}\right)^k = \left(\frac{t^2\|C\|}{4}\right)^{N+1} \sum_{k=N+1}^{\infty} \frac{1}{k!} \left(\frac{t^2\|C\|}{4}\right)^{k-N-1}$$
$$\leqslant \left(\frac{t^2\|C\|}{4}\right)^{N+1} \sum_{k=N+1}^{\infty} \frac{1}{(k-N-1)!} \left(\frac{t^2\|C\|}{4}\right)^{k-N-1} \leqslant \left(\frac{t^2\|C\|}{4}\right)^{N+1} e^{\,t^2\|C\|/4},$$

this implies that

$$\left\| \cos\left(t\sqrt{c-C}\right) - \sum_{k=0}^{N} \frac{1}{2^k k!} (\sqrt{ct^2})^{k+1} j_{k-1}(\sqrt{ct^2}) \left(\frac{1}{c}.C\right)^k \right\| \leqslant \frac{\pi}{N!} \min\left\{1, \left(\frac{t^2\|C\|}{4}\right)^{N+1}\right\} e^{\,t^2\|C\|/4}.$$

Furthermore,

$$\left\| \frac{\sin\left(t\sqrt{c-C}\right)}{\sqrt{c-C}} - t \sum_{k=0}^{N} \frac{1}{2^k k!} (\sqrt{ct^2})^k j_k(\sqrt{ct^2}) \left(\frac{1}{c}.C\right)^k \right\|$$
$$\leqslant |t| \sum_{k=N+1}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2})^k | j_k(\sqrt{ct^2})| \left(\frac{\|C\|}{c}\right)^k$$
$$\leqslant |t| \sum_{k=N+1}^{\infty} \frac{1}{2^k k!} (\sqrt{ct^2})^k \pi \frac{(\sqrt{ct^2})^k}{2^{k+1} k!} \left(\frac{\|C\|}{c}\right)^k$$
$$= \frac{\pi}{2} |t| \sum_{k=N+1}^{\infty} \frac{1}{(k!)^2} \left(\frac{t^2\|C\|}{4}\right)^k \leqslant \frac{\pi}{2(N+1)!} |t| \sum_{k=N+1}^{\infty} \frac{1}{k!} \left(\frac{t^2\|C\|}{4}\right)^k$$
$$\leqslant \frac{\pi}{2(N+1)!} |t| \min\left\{1, \left(\frac{t^2\|C\|}{4}\right)^{N+1}\right\} e^{\,t^2\|C\|/4}.$$

□

D. Proof of Theorem 3 (solutions of inhomogeneous wave equations)

We give a proof of Theorem 3.

Proof. In a first step, we note for $\lambda > 0$ that

$$\frac{\sin[(t-\tau)\sqrt{\lambda}\,]}{\sqrt{\lambda}} = \frac{\sin(t\sqrt{\lambda}\,)}{\sqrt{\lambda}} \cos(\tau\sqrt{\lambda}\,) - \cos(t\sqrt{\lambda}\,) \frac{\sin(\tau\sqrt{\lambda}\,)}{\sqrt{\lambda}}.$$

Then,

$$\frac{\sin\left((t-\tau)\sqrt{A}\right)}{\sqrt{A}} b(\tau) = \left\{ \frac{\sin\left(t\sqrt{A}\right)}{\sqrt{A}} \cos\left(\tau\sqrt{A}\right) - \cos\left(t\sqrt{A}\right) \frac{\sin\left(\tau\sqrt{A}\right)}{\sqrt{A}} \right\} b(\tau)$$
$$= \beta(t)\alpha(\tau)b(\tau) - \alpha(t)\beta(\tau)b(\tau)$$

for all $t, \tau \in \mathbb{R}$, where $\alpha, \beta : \mathbb{R} \to L(X, X)$ are defined by

$$\alpha(t) := \cos\left(t\sqrt{A}\right),\ \beta(t) := \frac{\sin\left(t\sqrt{A}\right)}{\sqrt{A}},$$

for every $t \in \mathbb{R}$. In the following, for $\xi \in D(A)$, we are going to use that the maps

$$(\mathbb{R} \to X, t \mapsto \alpha(t)\xi) \text{ and } (\mathbb{R} \to X, t \mapsto \beta(t)\xi)$$

are differentiable with derivatives

$$(\mathbb{R} \to X, t \mapsto -\beta(t)A\xi) \text{ and } (\mathbb{R} \to X, t \mapsto \alpha(t)\xi),$$

respectively. We note that, as a consequence of the spectral theorem for densely defined, self-adjoint linear operators in Hilbert spaces, that a, b are strongly continuous and that

$$\alpha(t)D(A) \subset D(A),\ \beta(t)D(A) \subset D(A),$$

for every $t \in \mathbb{R}$. Also for every $k \in U_{\mathbb{C}}(\sigma(A))^s$, $k(A)D(A) \subset D(A)$, and for $\xi \in D(A)$,

$$\|k(A)\xi\|_A^2 = \|k(A)\xi\|^2 + \|Ak(A)\xi\|^2 = \|k(A)\xi\|^2 + \|k(A)A\xi\|^2 \ \left[\leqslant \|k(A)\|_{\text{Op}}^2 \cdot \|\xi\|_A^2\right].$$

Hence a, b induce strongly continuous maps from $\mathbb{R}$ to X_A, which we indicate with the same symbols, and where $X_A := (D(A), \| \ \|_A)$. In addition, we note that the inclusion ι of X_A into X is continuous. In the next step, we observe for a strongly continuous $c : \mathbb{R} \to L(X_A, X_A)$ and a continuous $g : \mathbb{R} \to X_A$ that

$$\begin{aligned}\|c(t+h)g(t+h) - c(t)g(t)\|_A &= \|c(t+h)g(t+h) - c(t+h)g(t) + c(t+h)g(t) - c(t)g(t)\|_A \\ &= \|c(t+h)[g(t+h) - g(t)]_A + [c(t+h) - c(t)]g(t)\|_A \\ &\leqslant \|c(t+h)\| \cdot \|g(t+h) - g(t)\|_A + \|c(t+h)g(t) - c(t)g(t)\|_A\end{aligned}$$

and hence that $(\mathbb{R} \to X_A, t \mapsto c(t)g(t))$ is continuous as well as that

$$\left(\mathbb{R} \to X_A, t \mapsto \int_{I_t}^{A} c(\tau)g(\tau)d\tau\right),$$

where $\int^A$ denotes weak integration in X_A, is differentiable with derivative

$$(\mathbb{R} \to X_A, t \mapsto c(t)g(t)).$$

We conclude for every $t \in \mathbb{R}$ that

$$\begin{aligned}&\beta(t)\int_{I_t}^{A} \alpha(\tau)b(\tau)\,d\tau - \alpha(t)\int_{I_t}^{A} \beta(\tau)b(\tau)\,d\tau \\ &= \int_{I_t}^{A} [\beta(t)\alpha(\tau)b(\tau) - \alpha(t)\beta(\tau)b(\tau)]\,d\tau \\ &= \int_{I_t}^{A} \frac{\sin\left((t-\tau)\sqrt{A}\right)}{\sqrt{A}} b(\tau)\,d\tau = v(t).\end{aligned}$$

Furthermore, we observe for $c : \mathbb{R} \to L(X, X)$, $g : \mathbb{R} \to X$ such that $\text{Ran}(g) \subset D(A)$, $t \in \mathbb{R}$, and $h \in \mathbb{R}^*$ that

$$\begin{aligned}&\frac{1}{h}[c(t+h)g(t+h) - c(t)g(t)] \\ &= \frac{1}{h}[c(t+h)g(t+h) - c(t+h)g(t) + c(t+h)g(t) - c(t)g(t)] \\ &= c(t+h)\frac{1}{h}[g(t+h) - g(t)] + \frac{1}{h}[c(t+h) - c(t)]g(t) \\ &= c(t)\frac{1}{h}[g(t+h) - g(t)] + \frac{1}{h}[c(t+h)g(t) - c(t)g(t)] \\ &\quad + [c(t+h) - c(t)]\frac{1}{h}[g(t+h) - g(t)]\end{aligned}$$

and hence that

$$\begin{aligned}
&\frac{1}{h}\left[\alpha(t+h)g(t+h)-\alpha(t)g(t)\right]-\alpha(t)g'(t)+\beta(t)Ag(t)\\
&=\alpha(t)\left\{\frac{1}{h}\left[g(t+h)-g(t)\right]-g'(t)\right\}+\frac{1}{h}\left[\alpha(t+h)g(t)-\alpha(t)g(t)\right]+\beta(t)Ag(t)\\
&\quad+\left[\alpha(t+h)-\alpha(t)\right]\left\{\frac{1}{h}\left[g(t+h)-g(t)\right]-g'(t)\right\}+\left[\alpha(t+h)-\alpha(t)\right]g'(t),\\
&\frac{1}{h}\left[\beta(t+h)g(t+h)-\beta(t)g(t)\right]-\beta(t)g'(t)-\alpha(t)g(t)\\
&=\beta(t)\left\{\frac{1}{h}\left[g(t+h)-g(t)\right]-g'(t)\right\}+\frac{1}{h}\left[\beta(t+h)g(t)-\beta(t)g(t)\right]-\alpha(t)g(t)\\
&\quad+\left[\beta(t+h)-\beta(t)\right]\left\{\frac{1}{h}\left[g(t+h)-g(t)\right]-g'(t)\right\}+\left[\beta(t+h)-\beta(t)\right]g'(t).
\end{aligned}$$

This implies that

$$(\mathbb{R}\to X, t\mapsto \alpha(t)g(t)),\ (\mathbb{R}\to X,\ t\mapsto \beta(t)g(t))$$

are differentiable with derivatives

$$(\mathbb{R}\to X, t\mapsto \alpha(t)g'(t)-\beta(t)Ag(t)),\ (\mathbb{R}\to X,\ t\mapsto \beta(t)g'(t)+\alpha(t)g(t)),$$

respectively. Application of the latter to v gives for $t\in\mathbb{R}$,

$$\begin{aligned}
v'(t)&=\beta(t)\alpha(t)b(t)+\alpha(t)\int_{I_t}^{A}\alpha(\tau)b(\tau)\,d\tau-\alpha(t)\beta(t)b(t)+\beta(t)A\int_{I_t}^{A}\beta(\tau)b(\tau)\,d\tau\\
&=\alpha(t)\int_{I_t}^{A}\alpha(\tau)b(\tau)\,d\tau+\beta(t)\int_{I_t}\beta(\tau)Ab(\tau)\,d\tau\\
&=\alpha(t)\int_{I_t}^{A}\alpha(\tau)b(\tau)\,d\tau+\beta(t)\int_{I_t}^{A}\beta(\tau)Ab(\tau)\,d\tau,
\end{aligned}$$

where $\int$ denotes weak integration in X, and that

$$\begin{aligned}
v''(t)&=\alpha(t)\alpha(t)b(t)-\beta(t)A\int_{I_t}^{A}\alpha(\tau)b(\tau)\,d\tau+\beta(t)\beta(t)Ab(t)+\alpha(t)\int_{I_t}\beta(\tau)Ab(\tau)\,d\tau\\
&=\alpha(t)\alpha(t)b(t)+\beta(t)\beta(t)Ab(t)-\beta(t)A\int_{I_t}^{A}\alpha(\tau)b(\tau)\,d\tau+\alpha(t)A\int_{I_t}^{A}\beta(\tau)b(\tau)\,d\tau\\
&=b(t)-Av(t).
\end{aligned}$$

□

VII. CONCLUSION

Our result that the governing operator is a bounded function of the classical local operator for scalar-valued functions should be generalizable to vector-valued case. Our notable result that the governing operator A_C of the peridynamic wave equation is a bounded function of the classical governing operator has far reaching consequences. It enables the comparison of peridynamic solutions to those of classical elasticity. The remarkable implication is that it opens the possibly of defining peridynamic-type operators on bounded domains as functions of the corresponding classical operator. Since the classical operator is defined through *local* boundary conditions, the functions inherit this knowledge. This observation opens a gateway to incorporate local boundary conditions into nonlocal theories, which has vital implications for numerical treatment of nonlocal problems. This is the subject of our companion papers.[2,3]

We expect that the expansions in Theorems 22 and 24 can be used for obtaining the large time asymptotic of solutions of the nonlocal wave equation. In the classical case, as expected, we observe the propagation of waves along characteristics. In the nonlocal case, we observe oscillatory recurrent wave separation. We think that this phenomenon is worth investigating. On the other hand, we observe that discontinuity remains stationary in the nonlocal case, whereas, it is well-known that discontinuities propagate along characteristics. We hold that this fundamental difference is one of the most distinguishing features of PD. In conclusion, we believe that we added valuable tools to the arsenal of methods to analyze nonlocal problems.

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[70] Solution operators, $\cos(t\sqrt{A}\,)$ and $\sin(t\sqrt{A})/\sqrt{A}$, denote operators corresponding to unique extensions to entire holomorphic functions. A pedantic notation would be $\left[\overline{\cos\left(t\sqrt{}\right)}\Big|_{\sigma(A)}\right](A)$ and $\left[\overline{\frac{\sin\left(t\sqrt{}\right)}{\sqrt{}}}\Big|_{\sigma(A)}\right](A)$, respectively. For brevity, we prefer to use the former.
[71] $U^s_{\mathbb{C}}(\sigma(A), \mathbb{C})$ denotes the space of bounded complex-valued functions on $\sigma(A)$ that are strongly measurable in the sense that they are everywhere on $\sigma(A)$ limit of a sequence of step functions.