NEW ASYMMETRIC GENERALIZATIONS OF THE FILBERT AND LILBERT MATRICES

EMRAH KILIÇ, SIBEL KOPARAL, AND NEŞE ÖMÜR

Abstract. Two new asymmetric generalizations of the Filbert and Lilbert matrices constructed by the products of two Fibonacci and Lucas numbers are considered, with additional parameters settings. Explicit formulæ are derived for the LU-decompositions and their inverses.

1. Introduction

Let $\{U_n\}$ and $\{V_n\}$ be generalized Fibonacci and Lucas sequences, respectively, whose the Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1 + q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

When $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = (1-\sqrt{5})/(1+\sqrt{5})$), the sequence $\{U_n\}$ is reduced to the Fibonacci sequence $\{F_n\}$ and the sequence $\{V_n\}$ is reduced to the Lucas sequence $\{L_n\}$.

Throughout this paper we shall use the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$.

In the current literature, there are many interesting and useful combinatorial matrices constructed via the binomial coefficients, the Gaussian q-binomial coefficients or the well-known integer sequences such as natural numbers, the Fibonacci and Lucas numbers, etc. These matrices are chosen as Hankel matrix, Toeplitz matrix, tridiagonal matrix or any other special matrices. For these combinatorial matrices and their properties, we refer to the works (see [1-24]).

Now we recall some well-known combinatorial matrices from the current literature:

- Chu and Di Claudio [4] studied the matrix $\left[\frac{(a)_{j+\lambda_i}}{(c)_{j+\lambda_i}}\right]_{0\le i,j\le n}$, where a, c and $\{\lambda_i\}_{i=0}^n$ are complex numbers, and $(x)_n$ is shifted factorial of order n by

  $$(x)_0 = 1 \text{ and } (x)_n = x(x+1)\dots(x+n-1) \quad \text{for} \quad n = 1, 2, \dots.$$

  They also presented some variants of above matrix.
- For nonnegative integer g, Zhou and Zhaolin [23] studied the g-circulant matrices whose elements consist of the Fibonacci and the Lucas numbers, separately.
- Hilbert matrix $\mathcal{H} = [h_{ij}]$ is defined with entries

  $$h_{ij} = \frac{1}{i+j-1}.$$
- As an analogue of the Hilbert matrix, Richardson [22] defined the Filbert matrix $\mathcal{F} = [f_{ij}]$ with entries

  $$f_{ij} = \frac{1}{F_{i+j-1}}.$$
- In [8], Kılıç and Prodinger studied the generalized Filbert matrix $\mathfrak{F}$ with entries $\frac{1}{F_{i+j+r}}$, where $r \ge -1$ is an integer parameter.
- After this, Prodinger [20] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters by taking its entries as $\frac{x^i y^j}{F_{\lambda(i+j)+r}}$.

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- Kılıç and Prodinger [9] gave a further generalization of the generalized Filbert matrix $\mathcal{F}$ by defining the matrix $\mathcal{Q}$ with entries h_{ij} as follows
$$h_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1}\dots F_{i+j+r+k-1}},$$
where $r \geq -1$ and $k \geq 1$ are integer parameters.
- Recently, Kılıç and Prodinger introduced two new variations of the Filbert matrix $\mathcal{F}$, and define the matrices $\mathcal{G}$ and $\mathcal{L}$ with entries g_{ij} and t_{ij} by
$$g_{ij} = \frac{F_{\lambda(i+j)+r}}{F_{\lambda(i+j)+s}} \quad \text{and} \quad t_{ij} = \frac{L_{\lambda(i+j)+r}}{L_{\lambda(i+j)+s}},$$
where s, r and λ are integer parameters such that $s \neq r$, and $s \geq -1$ and $\lambda \geq 1$. This was the first nontrivial instance where the numerator of the entries is *not* equal to zero.
- Recently, Kılıç and Arıkan [18] define the matrices M and T as *nonlinear* generalizations of Filbert and Lilbert (Lucas-Hilbert) matrices with indices in geometric progression with entries
$$M_{ij} = \frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}} \text{ and } T_{ij} = \frac{1}{V_{\lambda(i+r)^k+\mu(j+s)^m+c}},$$
where U_n and V_n are nth generalized Fibonacci and Lucas numbers, resp. When $k = m = 1$, their results cover all Filbert-like matrices outside the matrices whose entries are consist of the products of the Fibonacci or Lucas numbers.
- Much recently, Kılıç and Prodinger [11] go one step further, by allowing an *asymmetric growth of indices*. They, however, confine theirselves to $k = 1$; for this instance, the inverse matrix also enjoys nice closed form entries, which is no longer true for $k \geq 2$. To be more specific, they introduce four generalizations of the Filbert matrix $\mathcal{F}$, and define the matrices $\mathcal{T}$, $\mathcal{M}$, $\mathcal{H}$ and $\mathcal{B}$ with entries by
$$t_{ij} = \frac{1}{F_{\lambda i+\mu j+r}},\ m_{ij} = \frac{F_{\lambda i+\mu j+r}}{F_{\lambda i+\mu j+s}},\ h_{ij} = \frac{1}{L_{\lambda i+\mu j+r}} \text{ and } b_{ij} = \frac{L_{\lambda i+\mu j+r}}{L_{\lambda i+\mu j+s}},$$
respectively, where s, r, λ and μ are integer parameters such that $s \neq r$, and $s \geq -1$ and $\lambda, \mu \geq 1$. The authors prove to their results, they couldn't use the q-Zeilberger algorithm because the summand they need are not q-hypergometric. So they use the backward induction to prove their claims.

In the works summarized above, the authors derived explicit formulæ for the LU-decomposition, their inverses, and the Cholesky factorization. Of course, because of these asymmetric entries, one cannot get a Cholesky decomposition anymore.

As mentioned above, there are various kinds of generalizations of the Filbert or Lilbert matrices. Some of them are based on increasing the number of Fibonacci or Lucas terms multiplied, see [9, 12]. On the other hand, an another kind generalization is to degenerate the symmetry in indices, see [11].

In this paper, we combine the generalization ideas of the works [9, 11, 12] and then continue to obtain "nice" generalizations of the Filbert and Lilbert matrices and then now we will introduce two generalizations of the Filbert matrix $\mathcal{F}$, and define the matrices $\mathcal{W}$ and $\mathcal{Z}$ by taking into products of the Fibonacci and Lucas numbers in the denominator as well as again allowing an *asymmetric growth of indices* with entries by for any positive integers i and j,
$$w_{ij} = \frac{1}{F_{\lambda i+\mu j+r}L_{\lambda i-\mu j+s}}$$
where r,s, λ and μ are integers such that $\lambda i + \mu j + r \neq 0$, and,
$$z_{ij} = \frac{1}{L_{\lambda i+\mu j+r}L_{\lambda i-\mu j+s}}$$
for any integers r,s, λ and μ, respectively.

Thus we rewrite the entries of the matrices $\mathcal{W}$ and $\mathcal{Z}$ in the q-form :
$$\begin{aligned} w_{ij} &= (-1)^{\lambda i}\,\mathbf{i}^{-r-s+1}q^{\lambda i+\frac{r+s-1}{2}}\frac{(1-q)}{(1-q^{\lambda i+\mu j+r})(1+q^{\lambda i-\mu j+s})}, \\ z_{ij} &= (-1)^{\lambda i}\,\mathbf{i}^{-r-s}q^{\lambda i+\frac{r+s}{2}}\frac{1}{(1+q^{\lambda i+\mu j+r})(1+q^{\lambda i-\mu j+s})}, \end{aligned}$$

respectively.

We will derive explicit formulæ for the LU-decompositions of matrices $\mathcal{W}$ and $\mathcal{Z}$, and their inverse. Similarly to the results of [8-18,20-21], the sizes of the matrices do not really matter, and they can be thought as infinite matrices and we may restrict it whenever necessary to the first N rows resp. columns and write $\mathcal{W}_N$ and $\mathcal{Z}_N$. All the identities we will obtain hold for general q, and results about Fibonacci and Lucas numbers come out as corollaries for the special choice of q.

Firstly, we will present all the results related to the matrices. Then we will indicate some proofs related to the matrices.

As an illustration, we write out the Fibonacci-Lucas cases for some $\lambda, \mu \in \{2, 1\}$ and $r = 3$, $s = 1$.

The important part is to *find* the explicit forms. This was done by experiments with a computer algebra system and spotting patterns. This becomes increasingly complicated when more and more new parameters are introduced, as the guessing only works for fixed choices of the parameters, and one needs to vary them as well.

Once one *knows* how the entries look like, proofs are by reducing sums to single terms. For this, the q-Zeilberger algorithm is a handy tool. However for the matrices $\mathcal{W}$ and $\mathcal{Z}$, the present versions of the q-Zeilberger algorithm do not work, and we have to *simulate* it by noticing that the relevant sums are Gosper-summable. To do this, some more guessing (with an additional parameter) is required. Consequently, since all these proofs are routine and somewhat tedious, we only present two typical examples.

2. The Matrix $\mathcal{W}$

In this section, we will obtain the LU-decomposition $\mathcal{W} = L.U$:

Theorem 1. *For $1 \le d \le n$, we have*

$$L_{n,d} = (-q)^{\lambda(n-d)} \frac{\left(q^\lambda; q^\lambda\right)_{n-1} \left(q^{\lambda d+\mu+r}; q^\mu\right)_d \left(-q^{\lambda d-\mu+s}; q^{-\mu}\right)_d \left(-q^{\lambda(n+d-1)+r+s}; q^{-\lambda}\right)_{n-d}}{\left(q^\lambda; q^\lambda\right)_{n-d} \left(q^\lambda; q^\lambda\right)_{d-1} \left(q^{\lambda n+\mu+r}; q^\mu\right)_d \left(-q^{\lambda n-\mu+s}; q^{-\mu}\right)_d \left(-q^{\lambda n+r+s}; q^{-\lambda}\right)_{n-d}}.$$

Fibonacci Corollary for $\lambda = 2$, $r = 3$ and $\mu = s = 1$:

Corollary 1. *For $1 \le d \le n$, we have*

$$L_{n,d} = \frac{\left(\prod_{t=1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{d} F_{t+2d+3}\right)\left(\prod_{t=1}^{d} L_{-t+2d+1}\right)\left(\prod_{t=1}^{n-d} L_{2(n+d-t+2)}\right)}{\left(\prod_{t=1}^{n-d} F_{2t}\right)\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{d} F_{2n+t+3}\right)\left(\prod_{t=1}^{d} L_{2n-t+1}\right)\left(\prod_{t=1}^{n-d} L_{2(n-t+3)}\right)}.$$

Theorem 2. *For $1 \le d \le n$, we have*

$$\begin{aligned} U_{d,n} &= \mathbf{i}^{-2\lambda d-r-s+1} q^{\binom{d+1}{2}\lambda+n(1-d)\mu+ds+\frac{r-s-1}{2}}(1-q) \\ &\times \frac{\left(q^\lambda; q^\lambda\right)_{d-1} \left(q^\mu; q^\mu\right)_{n-1} \left(-q^{\mu(n+1)+r-s}; q^\mu\right)_{d-1} \left(-q^{\lambda(d+1)+r+s}; q^\lambda\right)_{d-1}}{\left(q^\mu; q^\mu\right)_{n-d} \left(q^{\mu n+\lambda+r}; q^\lambda\right)_d \left(-q^{-\mu n+\lambda+s}; q^\lambda\right)_d \left(-q^{\lambda d-\mu+s}; q^{-\mu}\right)_{d-1} \left(q^{\lambda d+\mu+r}; q^\mu\right)_{d-1}}. \end{aligned}$$

Fibonacci Corollary for $\lambda = 2$, $r = 3$ and $\mu = s = 1$:

Corollary 2. *For $1 \le d \le n$, we have*

$$U_{d,n} = (-1)^{n-dn+d^2-1} \frac{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_t\right)\left(\prod_{t=1}^{d-1} L_{n+t+2}\right)\left(\prod_{t=1}^{d-1} L_{2(d+t+2)}\right)}{\left(\prod_{t=1}^{n-d} F_t\right)\left(\prod_{t=1}^{d} F_{n+2t+3}\right)\left(\prod_{t=1}^{d-1} F_{2d+t+3}\right)\left(\prod_{t=1}^{d} L_{-n+2t+1}\right)\left(\prod_{t=1}^{d-1} L_{2d-t+1}\right)}.$$

Theorem 3. *For $N \ge 1$, we have*

$$\begin{aligned} \det \mathcal{W}_N &= \mathbf{i}^{(-r-s+1)N} q^{\left(\frac{r-s-1}{2}\right)N} (1-q)^N \prod_{d=1}^{N} (-1)^{\lambda d} q^{\binom{d+1}{2}\lambda+d(1-d)\mu+ds} \\ &\times \frac{\left(q^\lambda; q^\lambda\right)_{d-1} \left(q^\mu; q^\mu\right)_{d-1} \left(-q^{\mu(d+1)+r-s}; q^\mu\right)_{d-1} \left(-q^{\lambda(d+1)+r+s}; q^\lambda\right)_{d-1}}{\left(q^{\mu d+\lambda+r}; q^\lambda\right)_d \left(-q^{-\mu d+\lambda+s}; q^\lambda\right)_d \left(-q^{\lambda d-\mu+s}; q^{-\mu}\right)_{d-1} \left(q^{\lambda d+\mu+r}; q^\mu\right)_{d-1}}. \end{aligned}$$

Fibonacci Corollary for $\lambda=2$, $r=3$ and $\mu=s=1$:

Corollary 3. *For $N\geq 1$, we have*

$$\det \mathcal{W}_N = (-1)^{\binom{N+2}{2}-1}\prod_{d=1}^{N}\frac{\left(\prod_{t=1}^{d-1}F_{2t}\right)\left(\prod_{t=1}^{d-1}F_{t}\right)\left(\prod_{t=1}^{d-1}L_{d+t+2}\right)\left(\prod_{t=1}^{d-1}L_{2(d+t+2)}\right)}{\left(\prod_{t=1}^{d}L_{2t-d+1}\right)\left(\prod_{t=1}^{d-1}L_{2d-t+1}\right)\left(\prod_{t=1}^{d}F_{d+2t+3}\right)\left(\prod_{t=1}^{d-1}F_{2d+t+3}\right)}.$$

Theorem 4. *For $1\leq d\leq n$, we have*

$$L_{n,d}^{-1} = (-1)^{(\lambda-1)(d-n)}\,q^{\lambda\binom{n-d+1}{2}}\left(1+q^{2\lambda d+r+s}\right)$$
$$\times\frac{\left(q^{\lambda};q^{\lambda}\right)_{n-1}\left(q^{\lambda d+\mu+r};q^{\mu}\right)_{n-1}\left(-q^{\lambda d-\mu+s};q^{-\mu}\right)_{n-1}\left(-q^{\lambda(n+1)+r+s};q^{\lambda}\right)_{n-1}}{\left(q^{\lambda};q^{\lambda}\right)_{n-d}\left(q^{\lambda};q^{\lambda}\right)_{d-1}\left(q^{\lambda n+\mu+r};q^{\mu}\right)_{n-1}\left(-q^{\lambda n-\mu+s};q^{-\mu}\right)_{n-1}\left(-q^{\lambda(n+d)+r+s};q^{-\lambda}\right)_{n}}.$$

Fibonacci Corollary for $\lambda=2$, $r=3$ and $\mu=s=1$:

Corollary 4. *For $1\leq d\leq n$, we have*

$$L_{n,d}^{-1} = (-1)^{d^2+n^2}\,L_{4(d+1)}\frac{\left(\prod_{t=1}^{n-1}F_{2t}\right)\left(\prod_{t=1}^{n-1}F_{2d+t+3}\right)\left(\prod_{t=1}^{n-1}L_{2d-t+1}\right)\left(\prod_{t=1}^{n-1}L_{2(n+t+2)}\right)}{\left(\prod_{t=1}^{n-d}F_{2t}\right)\left(\prod_{t=1}^{d-1}F_{2t}\right)\left(\prod_{t=1}^{n-1}F_{2n+t+3}\right)\left(\prod_{t=1}^{n-1}L_{2n-t+1}\right)\left(\prod_{t=1}^{n}L_{2(n+d-t+3)}\right)}.$$

Theorem 5. *For $1\leq d\leq n$, we have*

$$U_{d,n}^{-1} = (-1)^{d+(\lambda-1)n+1}\,\mathbf{i}^{r+s+1}q^{(\mu-\lambda)\binom{n+1}{2}+\mu\binom{d-1}{2}-\mu-sn+\frac{s-r+1}{2}}\frac{\left(1+q^{2\mu d+r-s}\right)}{(1-q)}$$
$$\times\frac{\left(q^{\mu d+\lambda+r};q^{\lambda}\right)_{n-1}\left(q^{\lambda n+\mu+r};q^{\mu}\right)_{n}\left(-q^{-\mu d+\lambda+s};q^{\lambda}\right)_{n-1}\left(-q^{\lambda n-\mu+s};q^{-\mu}\right)_{n}}{\left(q^{\lambda};q^{\lambda}\right)_{n-1}\left(q^{\mu};q^{\mu}\right)_{n-d}\left(q^{\mu};q^{\mu}\right)_{d-1}\left(-q^{\mu(d+1)+r-s};q^{\mu}\right)_{n}\left(-q^{\lambda(n+1)+r+s};q^{\lambda}\right)_{n-1}}.$$

Fibonacci Corollary for $\lambda=2$, $r=3$ and $\mu=s=1$:

Corollary 5. *For $1\leq d\leq n$, we have*

$$U_{d,n}^{-1} = (-1)^{\binom{d}{2}-\binom{n+1}{2}+1}\,L_{2(d+1)}\frac{\left(\prod_{t=1}^{n-1}F_{d+2t+3}\right)\left(\prod_{t=1}^{n}F_{2n+t+3}\right)\left(\prod_{t=1}^{n-1}L_{2t-d+1}\right)\left(\prod_{t=1}^{n}L_{2n-t+1}\right)}{\left(\prod_{t=1}^{n-1}F_{2t}\right)\left(\prod_{t=1}^{n-d}F_{t}\right)\left(\prod_{t=1}^{d-1}F_{t}\right)\left(\prod_{t=1}^{n}L_{d+t+2}\right)\left(\prod_{t=1}^{n-1}L_{2(n+t+2)}\right)}.$$

Theorem 6. *For $1\leq i,j\leq n$, we have*

$$\left(\left(\mathcal{W}_n\right)^{-1}\right)_{i,j} = (-1)^{j(\lambda-1)+i+1}\,\mathbf{i}^{r+s+1}q^{\binom{j}{2}\lambda+\mu\binom{i-1}{2}+\frac{s-r+1}{2}-\mu}\frac{\left(1+q^{2\lambda j+r+s}\right)\left(1+q^{2\mu i+r-s}\right)}{(1-q)\left(q^{\mu};q^{\mu}\right)_{i-1}\left(q^{\lambda};q^{\lambda}\right)_{j-1}}$$
$$\times\sum_{\max\{i,j\}\leq h\leq n}q^{\binom{h+1}{2}\mu-h(s+j\lambda)}\left(1-q^{h(\lambda+\mu)+r}\right)\left(1+q^{h(\lambda-\mu)+s}\right)$$
$$\times\frac{\left(q^{\mu i+\lambda+r};q^{\lambda}\right)_{h-1}\left(-q^{-\mu i+\lambda+s};q^{\lambda}\right)_{h-1}\left(q^{\lambda j+\mu+r};q^{\mu}\right)_{h-1}\left(-q^{\lambda j-\mu+s};q^{-\mu}\right)_{h-1}}{\left(q^{\mu};q^{\mu}\right)_{h-i}\left(-q^{\mu(i+1)+r-s};q^{\mu}\right)_{h}\left(q^{\lambda};q^{\lambda}\right)_{h-j}\left(-q^{\lambda(h+j)+r+s};q^{-\lambda}\right)_{h}}.$$

3. The Matrix $\mathcal{Z}$

Now we collect our results related to the matrix $\mathcal{Z}$. For convenience, we use the same letters L,U, but with a different meaning. We obtain the LU-decomposition $\mathcal{Z}=L.U$:

Theorem 7. *For $1\leq d\leq n$, we have*

$$L_{n,d} = (-q)^{\lambda(n-d)}\,\frac{\left(q^{\lambda(n+1)+r+s};q^{\lambda}\right)_{d-1}\left(q^{\lambda(n-1)};q^{-\lambda}\right)_{d-1}\left(-q^{\lambda d+\mu+r};q^{\mu}\right)_{d}\left(-q^{\lambda d-\mu+s};q^{-\mu}\right)_{d}}{\left(q^{\lambda(d+1)+r+s};q^{\lambda}\right)_{d-1}\left(q^{\lambda};q^{\lambda}\right)_{d-1}\left(-q^{\lambda n+\mu d+r};q^{-\mu}\right)_{d}\left(-q^{\lambda n-\mu+s};q^{-\mu}\right)_{d}}.$$

Fibonacci Corollary for $\lambda = 2$, $r = 3$ and $\mu = s = 1$:

Corollary 6. *For $1 \le d \le n$, we have*

$$L_{n,d} = \frac{\left(\prod_{t=1}^{d-1} F_{2(n-t)}\right)\left(\prod_{t=1}^{d-1} F_{2(n+t+2)}\right)\left(\prod_{t=1}^{d} L_{2d+t+3}\right)\left(\prod_{t=1}^{d} L_{2d-t+1}\right)}{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{d-1} F_{2(d+t+2)}\right)\left(\prod_{t=1}^{d} L_{2n+d-t+4}\right)\left(\prod_{t=1}^{d} L_{2n-t+1}\right)}.$$

Theorem 8. *For $1 \le d \le n$, we have*

$$U_{d,n} = \mathbf{i}^{-2\lambda d-r-s} q^{\lambda\binom{d+1}{2}+\mu n(1-d)+ds+\frac{r-s}{2}}$$
$$\times \frac{\left(q^\lambda;q^\lambda\right)_{d-1}\left(q^\mu;q^\mu\right)_{n-1}\left(q^{\mu(n+1)+r-s};q^\mu\right)_{d-1}\left(q^{\lambda(d+1)+r+s};q^\lambda\right)_{d-1}}{\left(q^\mu;q^\mu\right)_{n-d}\left(-q^{\lambda d-\mu+s};q^{-\mu}\right)_{d-1}\left(-q^{\mu n+\lambda+r};q^\lambda\right)_{d}\left(-q^{\lambda d+\mu+r};q^\mu\right)_{d-1}\left(-q^{-\mu n+\lambda+s};q^\lambda\right)_{d}}.$$

Fibonacci Corollary for $\lambda = 2$, $r = 3$ and $\mu = s = 1$:

Corollary 7. *For $1 \le d \le n$, we have*

$$U_{d,n} = (-1)^{d^2+n-dn-1}\, 5^{2(d-1)} \frac{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} F_{t}\right)\left(\prod_{t=1}^{d-1} F_{n+t+2}\right)\left(\prod_{t=1}^{d-1} F_{2(d+t+2)}\right)}{\left(\prod_{t=1}^{n-d} F_{t}\right)\left(\prod_{t=1}^{d-1} L_{2d-t+1}\right)\left(\prod_{t=1}^{d} L_{n+2t+3}\right)\left(\prod_{t=1}^{d-1} L_{2d+t+3}\right)\left(\prod_{t=1}^{d} L_{2t-n+1}\right)}.$$

Theorem 9. *For $N \ge 1$, we have*

$$\det \mathcal{Z}_N = \mathbf{i}^{(-r-s)N} q^{\left(\frac{r-s}{2}\right)N} \prod_{d=1}^{N} (-1)^{\lambda d}\, q^{\binom{d+1}{2}\lambda+d(1-d)\mu+ds}$$
$$\times \frac{\left(q^\lambda;q^\lambda\right)_{d-1}\left(q^\mu;q^\mu\right)_{d-1}\left(q^{\mu(d+1)+r-s};q^\mu\right)_{d-1}\left(q^{\lambda(d+1)+r+s};q^\lambda\right)_{d-1}}{\left(-q^{\mu d+\lambda+r};q^\lambda\right)_{d}\left(-q^{-\mu d+\lambda+s};q^\lambda\right)_{d}\left(-q^{\lambda d-\mu+s};q^{-\mu}\right)_{d-1}\left(-q^{\lambda d+\mu+r};q^\mu\right)_{d-1}}.$$

Fibonacci Corollary for $\lambda = 2$, $r = 3$ and $\mu = s = 1$:

Corollary 8. *For $N \ge 1$, we have*

$$\det \mathcal{Z}_N = (-1)^{\binom{N+2}{2}-1}\, 5^{N(N-1)} \prod_{d=1}^{N} \frac{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{d-1} F_{t}\right)\left(\prod_{t=1}^{d-1} F_{d+t+2}\right)\left(\prod_{t=1}^{d-1} F_{2d+2t+4}\right)}{\left(\prod_{t=1}^{d-1} L_{2d-t+1}\right)\left(\prod_{t=1}^{d} L_{d+2t+3}\right)\left(\prod_{t=1}^{d-1} L_{2d+t+3}\right)\left(\prod_{t=1}^{d} L_{2t-d+1}\right)}.$$

Theorem 10. *For $1 \le d \le n$, we have*

$$L_{n,d}^{-1} = (-1)^{(\lambda+1)(n-d)}\, q^{\lambda\binom{n-d+1}{2}} \left(1-q^{2\lambda d+r+s}\right)$$
$$\times \frac{\left(-q^{\lambda d+\mu+r};q^\mu\right)_{n-1}\left(-q^{\lambda d-\mu+s};q^{-\mu}\right)_{n-1}\left(q^\lambda;q^\lambda\right)_{n-1}\left(q^{\lambda(2n-1)+r+s};q^{-\lambda}\right)_{n-1}}{\left(-q^{\lambda n+\mu+r};q^\mu\right)_{n-1}\left(q^\lambda;q^\lambda\right)_{d-1}\left(q^\lambda;q^\lambda\right)_{n-d}\left(-q^{\lambda n-\mu+s};q^{-\mu}\right)_{n-1}\left(q^{\lambda(n+d)+r+s};q^{-\lambda}\right)_{n}}.$$

Fibonacci Corollary for $\lambda = 2$, $r = 3$ and $\mu = s = 1$:

Corollary 9. *For $1 \le d \le n$, we have*

$$L_{n,d}^{-1} = (-1)^{n^2+d^2}\, F_{4(d+1)} \frac{\left(\prod_{t=1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{n-1} L_{2d+t+3}\right)\left(\prod_{t=1}^{n-1} L_{2d-t+1}\right)\left(\prod_{t=1}^{n-1} F_{2(2n-t+2)}\right)}{\left(\prod_{t=1}^{d-1} F_{2t}\right)\left(\prod_{t=1}^{n-d} F_{2t}\right)\left(\prod_{t=1}^{n-1} L_{2n+t+3}\right)\left(\prod_{t=1}^{n-1} L_{2n-t+1}\right)\left(\prod_{t=1}^{n} F_{2(n+d-t+3)}\right)}.$$

Theorem 11. *For $1 \le d \le n$ and arbitrary integers μ, r, s such that $\left(q^{\mu(n+1)+r-s};q^\mu\right)_{d-1} \neq 0$, we have*

$$U_{d,n}^{-1} = (-1)^{d+(\lambda-1)n}\, \mathbf{i}^{r+s} q^{(\mu-\lambda)\binom{n+1}{2}+\mu\binom{d-1}{2}-\mu-sn-\frac{r-s}{2}} \left(1-q^{2\mu d+r-s}\right)$$
$$\times \frac{\left(-q^{\lambda n+\mu+r};q^\mu\right)_{n}\left(-q^{\mu d+\lambda+r};q^\lambda\right)_{n-1}\left(-q^{-\mu d+\lambda+s};q^\lambda\right)_{n-1}\left(-q^{\lambda n-\mu+s};q^{-\mu}\right)_{n}}{\left(q^\lambda;q^\lambda\right)_{n-1}\left(q^\mu;q^\mu\right)_{n-d}\left(q^\mu;q^\mu\right)_{d-1}\left(q^{\mu(d+1)+r-s};q^\mu\right)_{n}\left(q^{\lambda(n+1)+r+s};q^\lambda\right)_{n-1}}.$$

Fibonacci Corollary for $\lambda = 2$, $r = 3$ and $\mu = s = 1$:

Corollary 10. *For $1 \le d \le n$, we have*

$$U_{d,n}^{-1} = (-1)^{\binom{d}{2}-\binom{n+1}{2}+1}\, 5^{2(1-n)} F_{2(d+1)} \frac{\left(\prod_{t=1}^{n} L_{2n+t+3}\right)\left(\prod_{t=1}^{n-1} L_{d+2t+3}\right)\left(\prod_{t=1}^{n-1} L_{2t-d+1}\right)\left(\prod_{t=1}^{n} L_{2n-t+1}\right)}{\left(\prod_{t=1}^{n-1} F_{2t}\right)\left(\prod_{t=1}^{n-d} F_{t}\right)\left(\prod_{t=1}^{d-1} F_{t}\right)\left(\prod_{t=1}^{n} F_{d+t+2}\right)\left(\prod_{t=1}^{n-1} F_{2(n+t+2)}\right)}.$$

Theorem 12. *For $1 \le i, j \le n$, we have*

$$\begin{aligned}\left((\mathcal{Z}_n)^{-1}\right)_{i,j} &= (-1)^{i-j-j\lambda}\, \mathbf{i}^{r+s} q^{\lambda\binom{j}{2}+\mu\binom{i}{2}-i\mu-\frac{r-s}{2}} \frac{\left(1-q^{2\mu i+r-s}\right)\left(1-q^{2\lambda j+r+s}\right)}{(q^\mu;q^\mu)_{i-1}\,(q^\lambda;q^\lambda)_{j-1}}\\
&\times \sum_{\max\{i,j\}\le h\le n} q^{\mu\binom{h+1}{2}-h(s+j\lambda)}\left(1+q^{h(\lambda+\mu)+r}\right)\left(1+q^{h(\lambda-\mu)+s}\right)\left(q^{\lambda(2h-1)+r+s};q^{-\lambda}\right)_{h-1}\\
&\times \frac{\left(-q^{\mu i+\lambda+r};q^\lambda\right)_{h-1}\left(-q^{-\mu i+\lambda+s};q^\lambda\right)_{h-1}\left(-q^{\lambda j+\mu+r};q^\mu\right)_{h-1}\left(-q^{\lambda j-\mu+s};q^{-\mu}\right)_{h-1}}{(q^\lambda;q^\lambda)_{h-j}\,(q^\mu;q^\mu)_{h-i}\left(q^{\mu(i+1)+r-s};q^\mu\right)_h\left(q^{\lambda(h+1)+r+s};q^\lambda\right)_{h-1}\left(q^{\lambda(h+j)+r+s};q^{-\lambda}\right)_h}.\end{aligned}$$

4. Proof

It is mentioned as in the Introduction section, we give now two proofs about the matrix $\mathcal{W}$. We will not give proofs related with the matrix $\mathcal{Z}$. But we shall note that the proofs for the matrix $\mathcal{Z}$ will be similar to the matrix $\mathcal{W}$ We start with an introductory remark. For all the identities that we need to prove, experiments indicate that they are Gosper-summable. However, the entries that we encounter in our instances, do not qualify for the q-Zeilberger algorithm that we used in our earlier papers. Therefore, it was necessary to *guess* the relevant quantities; the justification is then complete routine. However, this guessing procedure is (with all the parameters involved) extremely time consuming, and so we confined ourselves to the demonstration of one such proof. We hope that extensions of the q-Zeilberger algorithm will be developed that fit our needs.

We deal now with

$$\sum_{d\le t\le m} L_{m,t} L_{t,d}^{-1}$$

and prove that it is 1 for $d = m$ (there is only one term in the sum) and 0 for $d > m$ since we have lower triangular matrices. So let us assume $m > d$. We will prove a general formula depending on an extra variable K:

$$\begin{aligned}\sum_{d\le t\le K} L_{m,t}L_{t,d}^{-1} &= (-1)^{K+d(\lambda-1)+\lambda m}\, q^{\lambda\left(\binom{K+1}{2}+\binom{d}{2}-m(K-1)-d\right)}\\
&\times \frac{\left(-q^{\lambda+r+s};q^\lambda\right)_d\left(1+q^{2\lambda d+r+s}\right)\left(-q^{-\lambda d+\mu-s};q^\mu\right)_K}{(q^\lambda;q^\lambda)_{d-1}\left(1+q^{\lambda(d+m)+r+s}\right)\left(1-q^{\lambda(m-d)}\right)\left(-q^{-\lambda m+\mu-s};q^\mu\right)_K}\\
&\times \frac{\left(q^{\lambda d+\mu+r};q^\mu\right)_K\left(q^\lambda;q^\lambda\right)_{m-1}\left(-q^{\lambda+r+s};q^\lambda\right)_{m+K}}{\left(q^{\lambda m+\mu+r};q^\mu\right)_K(q^\lambda;q^\lambda)_{m-K-1}(q^\lambda;q^\lambda)_{K-d}\left(-q^{\lambda+r+s};q^\lambda\right)_m\left(-q^{\lambda+r+s};q^\lambda\right)_{K+d}}.\end{aligned}$$

The formula we need follows from setting $K := m$. Note that the RHS of formula equals 0 when $K = m > d$ because of the term $(q^\lambda;q^\lambda)_{m-K-1}$ in the denominator of the third row. The proof of the formula is by induction. Clearly it is true for $K = d$, and the induction step amounts to show that

$$\sum_{d\le t\le K} L_{m,t}L_{t,d}^{-1} + L_{m,K+1}L_{K+1,d}^{-1} = \sum_{d\le t\le K+1} L_{m,t}L_{t,d}^{-1}.$$

Here consider the LHS of the claim

$$
\begin{aligned}
&\sum_{d\le t\le K} L_{m,t}L^{-1}_{t,d} + L_{m,K+1}L^{-1}_{K+1,d}\\
&= (-1)^{K+d(\lambda-1)+\lambda m}\, q^{\lambda\left(\binom{K+1}{2}+\binom{d}{2}-m(K-1)-d\right)}\\
&\times \frac{\left(1+q^{2\lambda d+r+s}\right)\left(q^{\lambda};q^{\lambda}\right)_{m-1}\left(-q^{\lambda+r+s};q^{\lambda}\right)_{d}}{\left(1+q^{\lambda(d+m)+r+s}\right)\left(1-q^{\lambda(m-d)}\right)\left(-q^{\lambda+r+s};q^{\lambda}\right)_{m}\left(q^{\lambda};q^{\lambda}\right)_{d-1}}\\
&\times \frac{\left(-q^{\lambda+r+s};q^{\lambda}\right)_{m+K}\left(-q^{-\lambda d+\mu-s};q^{\mu}\right)_{K}\left(q^{\lambda d+\mu+r};q^{\mu}\right)_{K}}{\left(q^{\lambda};q^{\lambda}\right)_{m-K-1}\left(q^{\lambda};q^{\lambda}\right)_{K-d}\left(-q^{\lambda+r+s};q^{\lambda}\right)_{K+d}\left(-q^{-\lambda m+\mu-s};q^{\mu}\right)_{K}\left(q^{\lambda m+\mu+r};q^{\mu}\right)_{K}}\\
&+ (-1)^{K+d(\lambda-1)+\lambda m+1}\, q^{\lambda\left(\binom{K+1}{2}+\binom{d}{2}+1-d-K(m-1)\right)}\\
&\times \left(1+q^{2\lambda d+r+s}\right)\left(1-q^{(\lambda+\mu)(K+1)+r}\right)\left(1+q^{(\mu-\lambda)(K+1)-s}\right)\frac{\left(q^{\lambda};q^{\lambda}\right)_{m-1}\left(-q^{\lambda+r+s};q^{\lambda}\right)_{d}}{\left(q^{\lambda};q^{\lambda}\right)_{d-1}\left(-q^{\lambda+r+s};q^{\lambda}\right)_{m}}\\
&\times \frac{\left(q^{\lambda d+\mu+r};q^{\mu}\right)_{K}\left(-q^{-\lambda d+\mu-s};q^{\mu}\right)_{K}\left(-q^{\lambda+r+s};q^{\lambda}\right)_{m+K}}{\left(q^{\lambda};q^{\lambda}\right)_{m-K-1}\left(q^{\lambda};q^{\lambda}\right)_{K+1-d}\left(q^{\lambda m+\mu+r};q^{\mu}\right)_{K+1}\left(-q^{-\lambda m+\mu-s};q^{\mu}\right)_{K+1}\left(-q^{\lambda+r+s};q^{\lambda}\right)_{d+K+1}},
\end{aligned}
$$

which, after some simple algebra and simplifications, gives us

$$\sum_{d\le t\le K+1} L_{m,t}L^{-1}_{t,d},$$

as claimed.

As a second proof, we will prove the result about LU-decomposition of the matrix.

First, we show that $\sum_t L_{d,t}U_{t,n}$ is indeed the matrix $\mathcal{W}$, that is,

$$\sum_{1\le t\le \min\{d,n\}} L_{d,t}U_{t,n} = (-1)^{\lambda d}\,\mathbf{i}^{-r-s+1}q^{\lambda d+\frac{r+s-1}{2}}\frac{(1-q)}{\left(1-q^{\lambda d+\mu n+r}\right)\left(1+q^{\lambda d-\mu n+s}\right)}.$$

We can assume without loss of generality that $n\ge d$ as well as we will prove a general formula depending on an extra variable K:

$$
\begin{aligned}
\sum_{K\le t\le d} L_{d,t}U_{t,n} &= (-1)^{\lambda d}\,\mathbf{i}^{-r-s+1}q^{\frac{r+s-1}{2}+d\lambda+n\mu}(1-q)\left(q^{\lambda};q^{\lambda}\right)_{d-1}\left(q^{\mu};q^{\mu}\right)_{n-1}\frac{1}{\left(1+q^{\lambda d-\mu n+s}\right)\left(1-q^{\lambda d+\mu n+r}\right)}\\
\times&\frac{q^{\left(\binom{K}{2}-(K-1)d\right)\lambda+\left(\binom{K}{2}-Kn\right)\mu}\left(-q^{(d+1)\lambda+r+s};q^{\lambda}\right)_{K-1}\left(-q^{(n+1)\mu+r-s};q^{\mu}\right)_{K-1}}{\left(q^{\lambda};q^{\lambda}\right)_{d-K}\left(q^{\mu};q^{\mu}\right)_{n-K}\left(-q^{-\mu n+\lambda+s};q^{\lambda}\right)_{K-1}\left(q^{\mu n+\lambda+r};q^{\lambda}\right)_{K-1}\left(q^{\lambda d+\mu+r};q^{\mu}\right)_{K-1}\left(-q^{-\lambda d+\mu-s};q^{\mu}\right)_{K-1}}.
\end{aligned}
$$

The formula we need follows from setting $K:=1$. The proof of the formula is by induction. Clearly it is true for $K=d$, and the induction step amounts to show that

$$\sum_{K-1\le t\le d} L_{d,t}U_{t,n} = L_{d,K-1}U_{K-1,n} + \sum_{K\le t\le d} L_{d,t}U_{t,n}.$$

By the induction hypothesis and and the definitions of the matrices L and U, we have the claim after some rearrangements.

Now we turn to the inverse matrix. Since

$$
\begin{aligned}
L^{-1}_{n,d} &= (-1)^{(\lambda-1)(d-n)}\, q^{\lambda\binom{n-d+1}{2}}\left(1+q^{2\lambda d+r+s}\right)\\
&\times \frac{\left(q^{\lambda};q^{\lambda}\right)_{n-1}\left(q^{\lambda d+\mu+r};q^{\mu}\right)_{n-1}\left(-q^{\lambda d-\mu+s};q^{-\mu}\right)_{n-1}\left(-q^{\lambda(n+1)+r+s};q^{\lambda}\right)_{n-1}}{\left(q^{\lambda};q^{\lambda}\right)_{n-d}\left(q^{\lambda};q^{\lambda}\right)_{d-1}\left(q^{\lambda n+\mu+r};q^{\mu}\right)_{n-1}\left(-q^{\lambda n-\mu+s};q^{-\mu}\right)_{n-1}\left(-q^{\lambda(n+d)+r+s};q^{-\lambda}\right)_{n}}
\end{aligned}
$$

and

$$U_{d,n}^{-1}=(-1)^{d+(\lambda-1)n+1}\,\mathbf{i}^{r+s+1}q^{(\mu-\lambda)\binom{n+1}{2}+\mu\binom{d-1}{2}-\mu-sn+\frac{s-r+1}{2}}\frac{\left(1+q^{2\mu d+r-s}\right)}{(1-q)}$$
$$\times\frac{\left(q^{\mu d+\lambda+r};q^{\lambda}\right)_{n-1}\left(q^{\lambda n+\mu+r};q^{\mu}\right)_{n}\left(-q^{-\mu d+\lambda+s};q^{\lambda}\right)_{n-1}\left(-q^{\lambda n-\mu+s};q^{-\mu}\right)_{n}}{\left(q^{\lambda};q^{\lambda}\right)_{n-1}\left(q^{\mu};q^{\mu}\right)_{n-d}\left(q^{\mu};q^{\mu}\right)_{d-1}\left(-q^{\mu(d+1)+r-s};q^{\mu}\right)_{n}\left(-q^{\lambda(n+1)+r+s};q^{\lambda}\right)_{n-1}},$$

we write

$$\left((\mathcal{W}_n)^{-1}\right)_{i,j}=\sum_h U_{i,h}^{-1}L_{h,j}^{-1}$$
$$=(-1)^{j(\lambda-1)+i+1}\,\mathbf{i}^{r+s+1}q^{\binom{j}{2}\lambda+\mu\binom{i-1}{2}+\frac{s-r+1}{2}-\mu}\frac{\left(1+q^{2\lambda j+r+s}\right)\left(1+q^{2\mu i+r-s}\right)}{(1-q)\left(q^{\mu};q^{\mu}\right)_{i-1}\left(q^{\lambda};q^{\lambda}\right)_{j-1}}$$
$$\times\sum_h q^{\binom{h+1}{2}\mu-h(s+j\lambda)}\left(1-q^{h(\lambda+\mu)+r}\right)\left(1+q^{h(\lambda-\mu)+s}\right)$$
$$\times\frac{\left(q^{\mu i+\lambda+r};q^{\lambda}\right)_{h-1}\left(-q^{-\mu i+\lambda+s};q^{\lambda}\right)_{h-1}\left(q^{\lambda j+\mu+r};q^{\mu}\right)_{h-1}\left(-q^{\lambda j-\mu+s};q^{-\mu}\right)_{h-1}}{\left(q^{\mu};q^{\mu}\right)_{h-i}\left(-q^{\mu(i+1)+r-s};q^{\mu}\right)_{h}\left(q^{\lambda};q^{\lambda}\right)_{h-j}\left(-q^{\lambda(h+j)+r+s};q^{-\lambda}\right)_{h}}.$$

The final formula as given in the theorem follows from some straightforward simplifications. Unfortunately, the sum cannot be evaluated in closed form.

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