A *NONLINEAR* GENERALIZATION OF THE FILBERT MATRIX AND ITS LUCAS ANALOGUE

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Abstract. In this paper, we present both a new generalization and an analogue of the Filbert matrix $\mathcal{F}$ by the means of the Fibonacci and Lucas numbers whose indices are in nonlinear form $\lambda (i+r)^k + \mu (j+s)^m + c$ for the positive integers λ, μ, k, m and the integers r, s, c. This will be the first example as nonlinear generalizations of the Filbert and Lilbert matrices. Furthermore we present q-versions of these matrices and their related results. We derive explicit formulæ for the inverse matrix, the LU-decomposition and the inverse matrices L^{-1}, U^{-1} as well as we present the Cholesky decomposition for all matrices.

1. Introduction and preliminaries

Define the second order linear recursive sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$, by

$$U_n = pU_{n-1} + U_{n-2} \text{ and } V_n = pV_{n-1} + V_{n-2}$$

with initials $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, resp.

Especially, when $p = 1$, the sequences $\{U_n\}$ and $\{V_n\}$ are reduced to the Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$, respectively. Also for the case $p = 2$, the sequence $\{U_n\}$ turns to the Pell sequence $\{P_n\}$.

The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1+q^n),$$

where $\alpha, \beta = \left(p \mp \sqrt{\Delta}\right)/2$ with $q = \beta/\alpha = -\alpha^2$ and $\Delta = p^2 + 4$, so that $\alpha = \mathbf{i}q^{-1/2}$, where $\mathbf{i} = \sqrt{-1}$.

The Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where $(x;q)_n$ stands for the q-Pochhammer symbol defined by

$$(x;q)_n = (1-x)(1-xq)\ldots\left(1-xq^{n-1}\right).$$

Note that

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

where $\binom{n}{k}$ is the usual binomial coefficient. For more detail, we refer to [1].

In the current literature, there are many interesting and useful combinatorial matrices. They are constructed via the binomial coefficients, the Gaussian q-binomial coefficients or the well-known integer sequences such as natural numbers, the Fibonacci and Lucas numbers, etc. (see [2–16]).

Now we recall some well-known combinatorial matrices from the current literature:

- Chu and Di Claudio [4] studied the matrix $\left[\frac{(a)_{j+\lambda_i}}{(c)_{j+\lambda_i}}\right]_{0 \leq i,j \leq n}$, where a and c are complex numbers, $\{\lambda_i\}_{i=0}^n$ are integers and $(x)_n$ is the shifted factorial of order n by

  $$(x)_0 = 1 \text{ and } (x)_n = x(x+1)\ldots(x+n-1) \quad \text{for } n = 1, 2, \ldots.$$

  They also presented some variants of the above matrix.

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- For nonnegative integer g, Zhou and Zhaolin [15] studied the g-circulant matrices whose elements consist of the Fibonacci and Lucas numbers, separately.
- Hilbert matrix $\mathcal{H} = [h_{ij}]$ is defined with entries
$$h_{ij} = \frac{1}{i+j-1}.$$
- As an analogue of the Hilbert matrix, Richardson [14] defined the Filbert matrix $\mathcal{F} = [f_{ij}]$ with entries
$$f_{ij} = \frac{1}{F_{i+j-1}},$$
where F_n stands for the nth Fibonacci number.
- In [7], Kılıç and Prodinger studied the generalized Filbert matrix with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter.
- After this, Prodinger [17] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters by taking its entries as $\frac{x^i y^j}{F_{\lambda(i+j)+r}}$, where $r \geq -1$ and $\lambda > 0$ are integers.
- Kılıç and Prodinger [18] gave a further generalization of the generalized Filbert Matrix $\mathcal{F}$ by defining the matrix $\mathcal{Q}$ with entries
$$q_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1}\dots F_{i+j+r+k-1}},$$
where $r \geq -1$ and $k \geq 0$ are integers.
- In a recent paper [19], Kılıç and Prodinger introduced the matrix $\mathcal{G}$ as a parametric generalization of the matrix $\mathcal{Q}$ by
$$g_{ij} = \frac{1}{F_{\lambda(i+j)+r}F_{\lambda(i+j+1)+r}\dots F_{\lambda(i+j+k-1)+r}},$$
where $r \geq -1$, $k \geq 0$ and $\lambda > 0$ are integer parameters.
- Much recently, Kılıç and Prodinger [20] gave four new generalizations of the Filbert matrix $\mathcal{F}$, by defining the matrices with the following entries
$$\frac{1}{F_{\lambda i+\mu j+r}},\ \frac{F_{\lambda i+\mu j+r}}{F_{\lambda i+\mu j+s}},\ \frac{1}{L_{\lambda i+\mu j+r}} \text{ and } \frac{L_{\lambda i+\mu j+r}}{L_{\lambda i+\mu j+s}},$$
where s, r, λ and μ are integer parameters such that $r, s \geq -1$, $s \neq r$ and $\lambda, \mu > 0$.
- As a Lucas analogue of the matrix $\mathcal{G}$, Kılıç and Prodinger [11] defined the matrix $\mathcal{W}$ by
$$w_{ij} = \frac{1}{L_{\lambda(i+j)+r}L_{\lambda(i+j+1)+r}\dots L_{\lambda(i+j+k-1)+r}}.$$

The authors of the all-above mentioned works have studied various properties of the given matrices such as LU and Cholesky decompositions, determinants, inverses etc. In many of them, firstly the authors converted the entries of the matrices into q-form and then proved all their claims in q-form by the means of the celebrated q-Zeilberger algorithm (see [21] for more details about the algorithm and also see [7,11,17–19,22] for its usage) or backward induction [20].

We would like to take attention of the readers to a point that the indices of the Fibonacci or Lucas numbers in the Filbert matrix and all its generalizations and analogues are in linear forms. For example, in the usual Filbert matrix $\mathcal{F} = \left[\frac{1}{F_{i+j-1}}\right]$, we see that the index is in the form $i+j-1$. Any *nonlinear form of the indices* has not been studied in anywhere according to our best literature knowledge. In this work, we present a new generalization of the Filbert matrix $\mathcal{F}$ as well as an analogue by the means of the Fibonacci and Lucas numbers whose indices will be in nonlinear form. This will be the first example in the literature.

Clearly, we will study the matrix A as a *nonlinear* generalization of Filbert matrix with indices in geometric progression given by
$$A_{ij} = \frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}},$$
where U_n is nth generalized Fibonacci number.

Moreover, as Lilbert (Lucas-Hilbert) analogue, we will study the matrix B with entries

$$B_{ij} = \frac{1}{V_{\lambda(i+r)^k+\mu(j+s)^m+c}},$$

where V_n is the nth generalized Lucas number.

It would be valuable to note that when $k = m = 1$, our results will cover all Filbert-like matrices except the matrices whose entries are consist of the products of the Fibonacci or Lucas numbers.

In Sections 2 and 3, we define the new generalizations of both the Filbert matrix and its Lucas analogue, Lilbert matrix, respectively. For all matrices will be studied, we derive explicit formulæ for the inverse matrix, the LU-decomposition and the inverse matrices L^{-1}, U^{-1} as well as we present the Cholesky decomposition. In Section 4, we only prove the results of Section 2. The results of Section 3 could be similarly proven. In Section 5, we give q-forms of the results of Sections 2 and 3 for an indeterminate q without proof. These results are more generalizations of the results given in Sections 2 and 3. For special values of q, one may obtain many special cases.

In general, for each section, the size of the matrix does not really matter except the results about inverse matrix, so that we may think about an infinite matrix A and restrict it whenever necessary to the first N rows resp. columns and use the notation A_N.

Throughout the paper, we assume that λ, μ, k and m are positive integers, r, s and c are any integers such that $\lambda\,(i+r)^k + \mu\,(j+s)^m + c > 0$ for all positive integers i and j.

2. A generalization of the Filbert matrix

In this section, for the matrix A, we give its inverse and LU-decomposition as well as we present its Cholesky decomposition when the matrix is symmetric, that is, the case $r = s$, $k = m$ and $\lambda = \mu$. Also we derive explicit formulæ for the matrices $L^{-1}$and U^{-1}.

We obtain the LU-decomposition $A = LU$:

Theorem 1. *For $i, j \geq 1$,*

$$L_{ij} = \frac{\left(\prod_{t=1}^{j} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{j} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}$$

and

$$U_{ij} = (-1)^{(\lambda+\mu)\binom{i}{2}+(\lambda r+\mu s+c)(i+1)} \frac{\left(\prod_{t=1}^{i-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{i-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{i-1} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{i} U_{\mu(j+s)^m+\lambda(t+r)^k+c}\right)}.$$

We also determine the inverses of the matrices L and U:

Theorem 2. *For $i, j \geq 1$,*

$$L^{-1}_{ij} = (-1)^{(\lambda+1)(i+j)+\lambda\binom{i-j+1}{2}} \frac{\left(\prod_{t=1}^{i-j-1} U_{\lambda(i+r)^k-\lambda(t+j+r)^k}\right)}{\left(\prod_{t=1}^{i-j-1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{i-1} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{i-1} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}$$

and

$$U_{ij}^{-1} = (-1)^{\lambda\binom{j+1}{2}+\mu\binom{i+1}{2}+i(\mu j+1)+j(\lambda+1)+(\lambda r+\mu s+c)(j+1)}$$
$$\times \frac{\left(\prod_{t=1}^{j-1} U_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{j} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{i-1} U_{\mu(i+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-i} U_{\mu(j+s+1-t)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}.$$

Now we give the inverse of the matrix A. This time it depends on the dimension, so we compute A_N^{-1}.

Theorem 3. *For* $1 \le i, j \le N$,

$$\left(A_N^{-1}\right)_{ij} = (-1)^{i+j+\lambda\binom{j+1}{2}+\mu\binom{i+1}{2}+N(\lambda j+\mu i+c+\lambda r+\mu s)+c+\lambda r+\mu s}$$
$$\times \frac{1}{U_{\lambda(j+r)^k+\mu(i+s)^m+c}\left(\prod_{t=1}^{i-1} U_{\mu(i+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}$$
$$\times \frac{\left(\prod_{t=1}^{N} U_{\lambda(t+r)^k+\mu(i+s)^m+c}\right)\left(\prod_{t=1}^{N} U_{\mu(t+s)^m+\lambda(j+r)^k+c}\right)}{\left(\prod_{t=1}^{N-i} U_{\mu(N+s+1-t)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{N-j} U_{\lambda(N+r+1-t)^k-\lambda(j+r)^k}\right)}.$$

Finally, we provide the Cholesky decomposition of the matrix A when it is symmetric, that is $r = s$, $k = m$ and $\lambda = \mu$.

Theorem 4. *For* $i, j \ge 1$,

$$C_{ij} = \frac{\prod_{t=1}^{j-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}}{\prod_{t=1}^{j} U_{\lambda(i+r)^k+\lambda(t+r)^k+c}} \sqrt{(-1)^{c(j+1)}\, U_{2\lambda(j+r)^k+c}}.$$

Note that when $k = m = 1$, $\lambda = \mu = 1$, $r = s = 0$ and $c = -1$ with $p = 1$, the matrix A is reduced to the well-known Filbert matrix $\mathcal{F}$ and so we obtain the results of [14]. Similarly, when $k = m = 1$ and $r = s = 0$ our results cover the results of [20]. In addition the results given in [7] could be obtained by choosing the suitable parameters. For the cases $k > 1$ or $m > 1$, our results are all new.

3. The Lucas analogue of the generalized Filbert matrix

In this section we give the LU-decomposition of the matrix B, the matrices L^{-1} and U^{-1}, the inverse matrix B^{-1} and the Cholesky decomposition of the matrix B when $r = s$, $k = m$ and $\lambda = \mu$, respectively.

Theorem 5. *For* $i, j \ge 1$,

$$L_{ij} = \frac{\left(\prod_{t=1}^{j} V_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{j} V_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}$$

and

$$U_{ij} = (-1)^{(\lambda+\mu)\binom{i}{2}+(\lambda r+\mu s+c+1)(i+1)}\, \Delta^{i-1}$$

$$\times \frac{\left(\prod_{t=1}^{i-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{i-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{i-1} V_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{i} V_{\mu(j+s)^m+\lambda(t+r)^k+c}\right)},$$

where Δ *is defined as before.*

Theorem 6. *For* $i, j \geq 1$,

$$L_{ij}^{-1} = (-1)^{(\lambda+1)(i+j)+\lambda\binom{i-j+1}{2}} \frac{\left(\prod_{t=1}^{i-j-1} U_{\lambda(i+r)^k-\lambda(t+j+r)^k}\right)}{\left(\prod_{t=1}^{i-j-1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{i-1} V_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{i-1} V_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}$$

and

$$U_{ij}^{-1} = (-1)^{\lambda\binom{j+1}{2}+\mu\binom{i+1}{2}+i(\mu j+1)+j(\lambda+1)+(\lambda r+\mu s+c+1)(j+1)} \Delta^{1-j}$$

$$\times \frac{\left(\prod_{t=1}^{j-1} V_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{j} V_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{i-1} U_{\mu(i+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-i} U_{\mu(j+s+1-t)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)},$$

where Δ *is defined as before.*

Theorem 7. *For* $1 \leq i, j \leq N$,

$$\left(B_N^{-1}\right)_{ij} = \frac{1}{\Delta^{N-1} V_{\lambda(j+r)^k+\mu(i+s)^m+c}} \frac{(-1)^{i+j+\lambda\binom{j+1}{2}+\mu\binom{i+1}{2}+N(\lambda j+\mu i)+(N+1)(c+\lambda r+\mu s+1)}}{\left(\prod_{t=1}^{i-1} U_{\mu(i+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{N} V_{\lambda(t+r)^k+\mu(i+s)^m+c}\right)\left(\prod_{t=1}^{N} V_{\mu(t+s)^m+\lambda(j+r)^k+c}\right)}{\left(\prod_{t=1}^{N-i} U_{\mu(N+s+1-t)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{N-j} U_{\lambda(N+r+1-t)^k-\lambda(j+r)^k}\right)},$$

where Δ *is defined as before.*

Theorem 8. *For* $i, j \geq 1$,

$$C_{ij} = \frac{\prod_{t=1}^{j-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}}{\prod_{t=1}^{j} V_{\lambda(i+r)^k+\lambda(t+r)^k+c}} \sqrt{(-1)^{(c+1)(j+1)} \Delta^{j-1} V_{2\lambda(j+r)^k+c}},$$

where Δ *is defined as before.*

Here note that when $k = m = 1$ and $r = s = 0$ the above results are reduced to the results of [20]. Similarly, when $k = m = 1$, $\lambda = \mu = 1$, $r = s = 0$ and $c = -1$ with $p = 1$ the matrix B is reduced to the usual Lilbert matrix. For the cases $k > 1$ or $m > 1$, our results are all new.

4. The Proofs

As mentioned in the introduction section, we will only give the proofs for the results of Section 2. The proofs of the results of Section 3 could be similarly done.

We need the following three lemmas for later use.

Lemma 1.

$$\sum_{d=K}^{\min(i,j)} (-1)^{(\lambda+\mu)\binom{d}{2}+(\lambda r+\mu s+c)(d+1)} U_{\lambda(d+r)^k+\mu(d+s)^m+c}$$
$$\times \frac{\left(\prod_{t=1}^{d-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{d-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{d} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{d} U_{\mu(j+s)^m+\lambda(t+r)^k+c}\right)}$$
$$= \frac{(-1)^{(\lambda+\mu)\binom{K}{2}+(\lambda r+\mu s+c)(K+1)}}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}} \frac{\left(\prod_{t=1}^{K-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-1} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m+\lambda(t+r)^k+c}\right)}.$$

Proof. We will use the backward induction method. For brevity, denote the sum and the summand term on the LHS of the just above claim by $\mathtt{SUM}_K^{(1)}$ and S_d, resp. First, assume that $i \geq j$ so when $K = j$ the claim is obvious. Similarly for the case $j > i$, claim is clear. The backward induction step amounts to show that

$$\mathtt{SUM}_{K-1}^{(1)} = \mathtt{SUM}_K^{(1)} + S_{K-1}.$$

By the definitions of $\mathtt{SUM}_K^{(1)}$ and S_{K-1}, consider the RHS of the above equality

$$\frac{(-1)^{(\lambda+\mu)\binom{K}{2}+(\lambda r+\mu s+c)(K+1)}}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}} \frac{\left(\prod_{t=1}^{K-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-1} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m+\lambda(t+r)^k+c}\right)}$$
$$+ (-1)^{(\lambda+\mu)\binom{K-1}{2}+(\lambda r+\mu s+c)K} U_{\lambda(K-1+r)^k+\mu(K-1+s)^m+c}$$
$$\times \frac{\left(\prod_{t=1}^{K-2} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-1} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m+\lambda(t+r)^k+c}\right)}$$
$$= \frac{(-1)^{(\lambda+\mu)\binom{K-1}{2}+(\lambda r+\mu s+c)K}}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}} \frac{\left(\prod_{t=1}^{K-2} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-1} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-1} U_{\mu(j+s)^m+\lambda(t+r)^k+c}\right)}$$
$$\times \Big[(-1)^{(\lambda+\mu)(K-1)+(\lambda r+\mu s+c)} U_{\lambda(i+r)^k-\lambda(K-1+r)^k} U_{\mu(j+s)^m-\mu(K-1+s)^m}$$
$$+ U_{\lambda(K-1+r)^k+\mu(K-1+s)^m+c} U_{\lambda(i+r)^k+\mu(j+s)^m+c}\Big].$$

By using the fact $U_n = (-1)^{n-1} U_{-n}$, the last expression in the bracket is rewritten as

$$(-1)^{\lambda(i+r)+\mu(j+s)+c} U_{\lambda(K-1+r)^k-\lambda(i+r)^k} U_{\mu(K-1+s)^m-\mu(j+s)^m} \\ + U_{\lambda(K-1+r)^k+\mu(K-1+s)^m+c} U_{\lambda(i+r)^k+\mu(j+s)^m+c}, \tag{4.1}$$

and by using the identity

$$U_m U_n = (-1)^{n+k} U_{m+k-n} U_k + U_{m+k} U_{n-k}, \tag{4.2}$$

for $m = \mu(j+s)^m + \lambda(K-1+r)^k + c$, $n = \lambda(i+r)^k + \mu(K-1+s)^m + c$ and $k = \mu(K-1+s)^m - \mu(j+s)^m$, the expression in (4.1) equals

$$U_{\lambda(i+r)^k+\mu(K-1+s)^m+c} U_{\mu(j+s)^m+\lambda(K-1+r)^k+c}.$$

Finally we write

$$\mathtt{SUM}_K^{(1)} + S_{K-1} = \frac{(-1)^{(\lambda+\mu)\binom{K-1}{2}+(\lambda r+\mu s+c)(K)}}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}} \times \frac{\left(\prod_{t=1}^{K-2} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m-\mu(t+s)^m}\right)}{\left(\prod_{t=1}^{K-2} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-2} U_{\mu(j+s)^m+\lambda(t+r)^k+c}\right)},$$

which completes the proof. □

Lemma 2.

$$\sum_{d=j}^{K} (-1)^{(\lambda+1)(d+j)+\lambda\binom{d-j+1}{2}} U_{\lambda(d+r)^k+\mu(d+s)^m+c} \frac{\left(\prod_{t=1}^{d-j-1} U_{\lambda(d+r)^k-\lambda(t+j+r)^k}\right)}{\left(\prod_{t=1}^{d-j-1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\ \times \frac{\left(\prod_{t=1}^{d-1} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(d+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{d-1} U_{\lambda(d+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{d} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)} \\ = \frac{(-1)^{\lambda\binom{K-j}{2}+(\lambda+1)(K+j)}}{U_{\lambda(i+r)^k-\lambda(j+r)^k}} \frac{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}.$$

Proof. Denote the sum and the summand term on the LHS of the claim just above by $\mathtt{SUM}_K^{(2)}$ and S_d, resp. By using induction, the case $K = j$ is obvious. So the induction step amounts to show that

$$\mathtt{SUM}_{K+1}^{(2)} = \mathtt{SUM}_K^{(2)} + S_{K+1}.$$

So $\mathtt{SUM}_K^{(2)} + S_{K+1}$ equals

$$\frac{(-1)^{\lambda\binom{K-j}{2}+(\lambda+1)(K+j)}}{U_{\lambda(i+r)^k-\lambda(j+r)^k}} \frac{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$+ (-1)^{(\lambda+1)(K+1+j)+\lambda\binom{K-j+2}{2}} U_{\lambda(K+1+r)^k+\mu(K+1+s)^m+c}$$

$$\times \frac{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K+1} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-j+1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)},$$

which, after some rearrangements, equals

$$\frac{(-1)^{\lambda\binom{K+1-j}{2}+(K+1+j)}}{U_{\lambda(i+r)^k-\lambda(j+r)^k}} \frac{\left(\prod_{t=1}^{K} U_{\lambda(i+r)^k-\lambda(t+r)^k}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K+1} U_{\lambda(i+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{K-j+1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$\times \Big[U_{\lambda(i+r)^k-\lambda(j+r)^k} U_{\lambda(K+1+r)^k+\mu(K+1+s)^m+c}$$

$$-U_{\lambda(i+r)^k+\mu(K+1+s)^m+c} U_{\lambda(K+1+r)^k-\lambda(j+r)^k}\Big].$$

Since

$$\begin{aligned}
&(-1)^{\lambda(K+j+1)} \Big[U_{\lambda(i+r)^k-\lambda(j+r)^k} U_{\lambda(K+1+r)^k+\mu(K+1+s)^m+c} \\
&-U_{\lambda(i+r)^k+\mu(K+1+s)^m+c} U_{\lambda(K+1+r)^k-\lambda(j+r)^k}\Big] \\
&= U_{\lambda(i+r)^k-\lambda(K+1+r)^k} U_{\lambda(j+r)^k+\mu(K+1+s)^m+c},
\end{aligned}$$

the claim follows. □

Lemma 3.

$$\sum_{d=\max(i,j)}^{K} (-1)^{i\mu d+\lambda dj+(\lambda r+\mu s+c)d} U_{\lambda(d+r)^k+\mu(d+s)^m+c} \frac{\left(\prod_{t=1}^{d-j-1} U_{\lambda(d+r)^k-\lambda(t+j+r)^k}\right)}{\left(\prod_{t=1}^{d-j-1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$\times \frac{\left(\prod_{t=1}^{d-1} U_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(d+r)^k-\lambda(t+r)^k}\right)}{\left(\prod_{t=1}^{d-i} U_{\mu(d+s+1-t)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{d-1} U_{\lambda(d+r)^k-\lambda(t+r)^k}\right)}$$

$$= \frac{(-1)^{K(\lambda j+\mu i+c+\lambda r+\mu s)}}{U_{\lambda(j+r)^k+\mu(i+s)^m+c}} \frac{\left(\prod_{t=1}^{K} U_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}.$$

Proof. Denote the sum on the LHS of the claim by $\mathtt{SUM}_K^{(3)}$. If $j \geq i$, the case $K = j$ easily follows. If $i > j$, then

$$\mathtt{SUM}_i^3 = \frac{(-1)^{i(\mu i+\lambda j+\lambda r+\mu s+c)}}{U_{\lambda(i+r)^k-\lambda(j+r)^k}} \frac{\left(\prod_{t=1}^{i} U_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{i-1} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{i-j-1} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}$$

$$= \frac{(-1)^{i(\mu i+\lambda j+\lambda r+\mu s+c)}}{U_{\lambda(j+r)^k+\mu(i+s)^m+c}} \frac{\left(\prod_{t=1}^{i} U_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{i} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{i-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)}.$$

So the first step of induction is complete. By using the next step of induction, we have

$$\begin{aligned}
\mathtt{SUM}^{(3)}_{K+1} &= \frac{(-1)^{K(\lambda j+\mu i+c+\lambda r+\mu s)}}{U_{\lambda(j+r)^k+\mu(i+s)^m+c}} \frac{\left(\prod_{t=1}^{K} U_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\
&\quad + (-1)^{(i\mu+\lambda j+\lambda r+\mu s+c)(K+1)}\, U_{\lambda(K+1+r)^k+\mu(K+1+s)^m+c} \\
&\quad \times \frac{\left(\prod_{t=1}^{K} U_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K+1-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K+1-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)},
\end{aligned}$$

which, after some simplifications, equals

$$\begin{aligned}
&\frac{(-1)^{(i\mu+\lambda j+\lambda r+\mu s+c)(K+1)}}{U_{\lambda(j+r)^k+\mu(i+s)^m+c}} \frac{\left(\prod_{t=1}^{K} U_{\mu(i+s)^m+\lambda(t+r)^k+c}\right)\left(\prod_{t=1}^{K} U_{\lambda(j+r)^k+\mu(t+s)^m+c}\right)}{\left(\prod_{t=1}^{K+1-i} U_{\mu(t+i+s)^m-\mu(i+s)^m}\right)\left(\prod_{t=1}^{K+1-j} U_{\lambda(t+j+r)^k-\lambda(j+r)^k}\right)} \\
&\times \Big[(-1)^{\lambda(j+r)+\mu(i+s)+c}\, U_{\mu(K+1+s)^m-\mu(i+s)^m} U_{\lambda(K+1+r)^k-\lambda(j+r)^k} \\
&+ U_{\lambda(K+1+r)^k+\mu(K+1+s)^m+c} U_{\lambda(j+r)^k+\mu(i+s)^m+c}\Big].
\end{aligned}$$

By using the identity (4.2) with appropriate parameters, the last expression in the bracket equals

$$U_{\mu(i+s)^m+\lambda(K+1+r)^k+c} U_{\lambda(j+r)^k+\mu(K+1+s)^m+c},$$

so the proof follows by induction. □

Now we shall give the proofs of the results of Section 2.

For the matrices L and L^{-1}, it is obvious $L_{ii}L^{-1}_{ii} = 1$. For $i > j$, by Lemma 2

$$\sum_{j\le d\le i} L_{id}L^{-1}_{dj} = \mathtt{SUM}^{(2)}_i = 0,$$

so we conclude

$$\sum_{j\le d\le i} L_{id}L^{-1}_{dj} = \delta_{ij},$$

where δ_{ij} is Kronecker delta, as desired. Here we omit the proof of $UU^{-1} = I$, it could be similarly done by constructing a proper lemma.

For the LU-decomposition, we have to prove that

$$\sum_{1\le d\le \min\{i,j\}} L_{id}U_{dj} = A_{ij}.$$

By Lemma 1, we obtain

$$\sum_{1\le d\le \min\{i,j\}} L_{id}U_{dj} = \mathtt{SUM}^{(1)}_1 = \frac{1}{U_{\lambda(i+r)^k+\mu(j+s)^m+c}},$$

which completes the proof.

For the inverse matrix A_N^{-1}, we use the fact $A_N^{-1} = U_N^{-1} L_N^{-1}$. Consider

$$\begin{aligned}\sum_{\max\{i,j\}\le d\le N} U_{id}^{-1} L_{dj}^{-1} &= \frac{(-1)^{\mu\binom{i+1}{2}+i+\lambda r+\mu s+c+j+\lambda\binom{j+1}{2}}}{\left(\prod_{t=1}^{i-1} U_{\mu(i+s)^m-\mu(t+s)^m}\right)\left(\prod_{t=1}^{j-1} U_{\lambda(j+r)^k-\lambda(t+r)^k}\right)} \mathtt{SUM}_N^{(3)} \\ &= \left(A_N^{-1}\right)_{ij}.\end{aligned}$$

Thus the proofs of the results of Section 2 are complete.

5. q-analogue of the generalized Filbert matrix

In this section, we present q-forms of the results of Sections 2 and 3. The results for the matrices A and B given previously come out as corollaries of the results of this section for the special choice of $q = \beta/\alpha$. We omit the proofs not to bore the readers, they could be similarly done by finding the q-analogues of Lemmas 1, 2 and 3. Also note that mechanic summation method or q-Zeilberger algorithm will not work here due to the non-hypergeometric summand terms.

We denote the q-analogues of the matrices A and B by $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A}_{ij} = (-1)^{-\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c-1\right]} q^{\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c-1\right]} \frac{1-q}{1-q^{\lambda(i+r)^k+\mu(j+s)^m+c}}$$

and

$$\mathcal{B}_{ij} = (-1)^{-\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c\right]} q^{\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c\right]} \frac{1}{1+q^{\lambda(i+r)^k+\mu(j+s)^m+c}},$$

respectively.

For later use, we will define a generalization of the q-Pochhammer symbol with two additional parameters in which one of them is in geometric progression as follows

$$(a;q)_n^{(r,k)} := \left(1-aq^{(1+r)^k}\right)\left(1-aq^{(2+r)^k}\right)\ldots\left(1-aq^{(n+r)^k}\right) = \prod_{t=1}^{n}\left(1-aq^{(t+r)^k}\right)$$

with $\left(a;q^{(r,k)}\right)_0 = 1$, where a is a real number, r is an integer and n,k are positive integers. As examples, we note that

$$\begin{aligned}(1;q)_n^{(0,2)} &= (1-q)\left(1-q^4\right)\ldots\left(1-q^{n^2}\right),\\ \left(a;q^2\right)_n^{(1,2)} &= \left(1-aq^8\right)\left(1-aq^{18}\right)\ldots\left(1-aq^{2(n+1)^2}\right),\\ (-q;q)_n^{(-1,3)} &= (1+q)\left(1+q^2\right)\left(1+q^9\right)\cdots\left(1+q^{(n-1)^3+1}\right),\\ \left(a;q^\lambda\right)_n^{(0,1)} &= \left(1-aq^\lambda\right)\left(1-aq^{2\lambda}\right)\ldots\left(1-aq^{n\lambda}\right) = \left(aq^\lambda;q^\lambda\right)_n.\end{aligned}$$

So the relation between the usual q-Pochhammer symbol and the general q-Pochhammer notation is

$$(x;q)_n = (x;q)_n^{(-1,1)}.$$

As the q-analogue of the results of Section 2, we present the following results related with the matrix $\mathcal{A}$ and its LU-factorization, Cholesky factorization as well as the matrices L^{-1}, U^{-1} and the inverse matrix $\mathcal{A}_N^{-1}$.

Theorem 9. *For the matrix $\mathcal{A}$ and $i,j \ge 1$,*

$$L_{ij} = q^{\frac{1}{2}\lambda\left[(i+r)^k-(j+r)^k\right]} (-1)^{\frac{1}{2}\lambda\left[(j+r)^k-(i+r)^k\right]} \frac{\left(q^{\lambda(j+r)^k+c};q^\mu\right)_j^{(s,m)} \left(q^{\lambda(i+r)^k};q^{-\lambda}\right)_{j-1}^{(r,k)}}{\left(q^{\lambda(i+r)^k+c};q^\mu\right)_j^{(s,m)} \left(q^{\lambda(j+r)^k};q^{-\lambda}\right)_{j-1}^{(r,k)}},$$

$$U_{ij} = q^{-\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c+1\right]+i\left[\lambda(i+r)^k+\mu(j+s)^m+c\right]} (-1)^{-\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c-1\right]}$$

$$\times \frac{(1-q)\left(q^{-\lambda(i+r)^k};q^\lambda\right)_{i-1}^{(r,k)}\left(q^{-\mu(j+s)^m};q^\mu\right)_{i-1}^{(s,m)}}{\left(q^{\lambda(i+r)+c};q^\mu\right)_{i-1}^{(s,m)}\left(q^{\mu(j+s)^m+c};q^\lambda\right)_{i}^{(r,k)}},$$

$$L_{ij}^{-1} = (-1)^{\lambda(i+j)+1+\frac{1}{2}\lambda\left[(i+r)^k-(j+r)^k\right]} q^{\lambda(j-i)\left[(j+r)^k-(i+r)^k\right]+\frac{1}{2}\lambda\left[(j+r)^k-(i+r)^k\right]}$$
$$\times \frac{\left(q^{-\lambda(i+r)^k};q^\lambda\right)_{i-j-1}^{(j+r,k)}\left(q^{\lambda(j+r)^k+c};q^\mu\right)_{i-1}^{(s,m)}\left(q^{\lambda(i+r)^k};q^{-\lambda}\right)_{j-1}^{(r,k)}}{\left(q^{-\lambda(j+r)^k};q^\lambda\right)_{i-j-1}^{(j+r,k)}\left(q^{\lambda(i+r)^k+c};q^\mu\right)_{i-1}^{(s,m)}\left(q^{\lambda(j+r)^k};q^{-\lambda}\right)_{j-1}^{(r,k)}},$$

$$U_{ij}^{-1} = (-1)^{\mu i+\lambda j+\lambda r+\mu s+c-\frac{1}{2}\left[\mu(i+s)^m+\lambda(j+r)^k+c+1\right]} q^{\left(\frac{1}{2}-j\right)\left[\mu(i+s)^m+\lambda(j+r)^k+c\right]+\frac{1}{2}}$$
$$\times \frac{\left(q^{\mu(i+s)^m+c};q^\lambda\right)_{j-1}^{(r,k)}\left(q^{\lambda(j+r)^k+c};q^\mu\right)_{j}^{(s,m)}}{(1-q)\left(q^{-\mu(i+s)^m};q^\mu\right)_{i-1}^{(s,m)}\left(q^{-\lambda(j+r)^k};q^\lambda\right)_{j-1}^{(r,k)}\left(q^{-\mu(i+s)^m};q^\mu\right)_{j-i}^{(i+s,m)}},$$

$$\left(\mathcal{A}_N^{-1}\right)_{ij} = (-1)^{\mu i+\lambda j+\lambda r+\mu s+c-\frac{1}{2}\left[\mu(i+s)^m+\lambda(j+r)^k+c+1\right]} q^{\left(\frac{1}{2}-N\right)\left[\mu(i+s)^m+\lambda(j+r)^k+c+1\right]+\frac{1}{2}}$$
$$\times \frac{\left(q^{\mu(i+s)^m+c};q^\lambda\right)_{N}^{(r,k)}\left(q^{\lambda(j+r)^k+c};q^\mu\right)_{N}^{(s,m)}}{(1-q)\left(1-q^{\lambda(j+r)^k+\mu(i+s)^m+c}\right)\left(q^{-\mu(i+s)^m};q^\mu\right)_{i-1}^{(s,m)}\left(q^{-\lambda(j+r)^k};q^\lambda\right)_{j-1}^{(r,k)}}$$
$$\times \frac{1}{\left(q^{-\mu(i+s)^m};q^\mu\right)_{N-i}^{(i+s,m)}\left(q^{-\lambda(j+r)^k};q^\lambda\right)_{N-j}^{(j+r,k)}}$$

and for $r=s$, $k=m$ and $\lambda=\mu$,

$$C_{ij} = (-1)^{\lambda\left[jr+i+\binom{j}{2}\right]+j+1+\frac{1}{2}\left[\lambda(i+r)^k-\lambda(j+r)^k+1-cj\right]}$$
$$\times q^{\frac{1}{2}\left[\lambda(j+r)^k-\lambda(i+r)^k+cj-1\right]+\lambda(i+r)^k j}\frac{\left(q^{-\lambda(i+r)^k};q^\lambda\right)_{j-1}^{(r,k)}}{\left(q^{\lambda(i+r)^k+c};q^\lambda\right)_{j}^{(r,k)}}$$
$$\times \sqrt{(1-q)\,q^{-\lambda(j+r)^k+\frac{1-c}{2}}\,(-1)^{\lambda(j+r)^k+c(j+1)+\frac{c-1}{2}}\left(1-q^{2\lambda(j+r)^k+c}\right)}.$$

Similarly, as the q-analogue of the results of Section 3, we present the following results.

Theorem 10. *For the matrix $\mathcal{B}$ and $i,j\geq 1$,*

$$L_{ij} = q^{\frac{1}{2}\lambda\left[(i+r)^k-(j+r)^k\right]}(-1)^{\frac{1}{2}\lambda\left[(j+r)^k-(i+r)^k\right]}$$
$$\times \frac{\left(-q^{\lambda(j+r)^k+c};q^\mu\right)_{j}^{(s,m)}\left(q^{\lambda(i+r)^k};q^{-\lambda}\right)_{j-1}^{(r,k)}}{\left(-q^{\lambda(i+r)^k+c};q^\mu\right)_{j}^{(s,m)}\left(q^{\lambda(j+r)^k};q^{-\lambda}\right)_{j-1}^{(r,k)}},$$

$$U_{ij} = q^{(i-1)\left[\lambda(i+r)^k+\mu(j+s)^m+c\right]+\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c\right]}(-1)^{i-1-\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c\right]}$$
$$\times \frac{\left(q^{-\lambda(i+r)^k};q^\lambda\right)_{i-1}^{(r,k)}\left(q^{-\mu(j+s)^m};q^\mu\right)_{i-1}^{(s,m)}}{\left(-q^{\lambda(i+r)+c};q^\mu\right)_{i-1}^{(s,m)}\left(-q^{\mu(j+s)^m+c};q^\lambda\right)_{i}^{(r,k)}},$$

$$L_{ij}^{-1} = (-1)^{\lambda(i-j)+1+\frac{1}{2}\lambda\left[(i+r)^k-(j+r)^k\right]} q^{\lambda(j-i)\left[(j+r)^k-(i+r)^k\right]+\frac{1}{2}\lambda\left[(j+r)^k-(i+r)^k\right]}$$
$$\times \frac{\left(q^{-\lambda(i+r)^k};q^\lambda\right)_{i-j-1}^{(j+r,k)}\left(-q^{\lambda(j+r)^k+c};q^\mu\right)_{i-1}^{(s,m)}\left(q^{\lambda(i+r)^k};q^{-\lambda}\right)_{j-1}^{(r,k)}}{\left(q^{-\lambda(j+r)^k};q^\lambda\right)_{i-j-1}^{(j+r,k)}\left(-q^{\lambda(i+r)^k+c};q^\mu\right)_{i-1}^{(s,m)}\left(q^{\lambda(j+r)^k};q^{-\lambda}\right)_{j-1}^{(r,k)}},$$

$$U_{ij}^{-1} = q^{\left(\frac{1}{2}-j\right)\left[\mu(i+s)^m+\lambda(j+r)^k+c\right]}(-1)^{j-1+\frac{1}{2}\left[\mu(i+s)^m+\lambda(j+r)^k+c\right]}$$
$$\times \frac{\left(-q^{\mu(i+s)^m+c};q^\lambda\right)_{j-1}^{(r,k)}\left(-q^{\lambda(j+r)^k+c};q^\mu\right)_j^{(s,m)}}{\left(q^{-\mu(i+s)^m};q^\mu\right)_{i-1}^{(s,m)}\left(q^{-\lambda(j+r)^k};q^\lambda\right)_{j-1}^{(r,k)}\left(q^{-\mu(i+s)^m};q^\mu\right)_{j-i}^{(i+s,m)}},$$

$$\left(\mathcal{B}_N^{-1}\right)_{ij} = q^{\left(\frac{1}{2}-N\right)\left[\lambda(j+r)^k+\mu(i+s)^m+c\right]}(-1)^{\lambda j+\mu i+c+\lambda r+\mu s+N+1-\frac{1}{2}\left[\lambda(j+r)^k+\mu(i+s)^m+c\right]}$$
$$\times \frac{\left(-q^{\mu(i+s)^m+c};q^\lambda\right)_N^{(r,k)}\left(-q^{\lambda(j+r)^k+c};q^\mu\right)_N^{(s,m)}}{\left(1+q^{\lambda(j+r)^k+\mu(i+s)^m+c}\right)\left(q^{-\mu(i+s)^m};q^\mu\right)_{N-i}^{(i+s,m)}\left(q^{-\lambda(j+r)^k};q^\lambda\right)_{N-j}^{(j+r,k)}}$$
$$\times \frac{1}{\left(q^{-\mu(i+s)^m};q^\mu\right)_{i-1}^{(s,m)}\left(q^{-\lambda(j+r)^k};q^\lambda\right)_{j-1}^{(r,k)}}$$

and for $r = s$, $k = m$ and $\lambda = \mu$,

$$C_{ij} = (-1)^{j+1+\lambda\binom{j}{2}+\lambda r(j-1)-\frac{1}{2}\left[\lambda(i+r)^k+\lambda(j+r)^k+cj\right]} q^{\frac{1}{2}\left[\lambda(i+r)^k+\lambda(j+r)^k+cj\right]+\lambda(j-1)(i+r)^k}$$
$$\times \frac{\left(q^{-\lambda(i+r)^k};q^\lambda\right)_{j-1}^{(r,k)}}{\left(-q^{\lambda(i+r)^k+c};q^\lambda\right)_j^{(r,k)}}\sqrt{q^{-\lambda(j+r)^k-\frac{c}{2}}(-1)^{\lambda(j+r)+(c+1)(j+1)+\frac{c}{2}}\left(1+q^{2\lambda(j+r)^k+c}\right)}.$$

6. Conclusion Remarks

Recall that the results of Sections 2 and 3 are the special consequences of Theorems 9 and 10 for the case $q = \beta/\alpha$, respectively. Similarly one could derive many different consequences of our results for each q value.

For example, if we define the matrix $\mathcal{P} = \left[\frac{1}{P_{\lambda(i+r)^k+\mu(j+s)^m+c}}\right]_{1\le i,j}$, where P_n stands for the nth Pell number, then it is enough to take $q = \left(1-\sqrt{2}\right)/\left(1+\sqrt{2}\right)$ in Theorem 9 to obtain related results for the matrix $\mathcal{P}$. Indeed from the LU-decomposition of the matrix $\mathcal{P}$, we could obtain that

$$\det \mathcal{P}_N = (-1)^{(\lambda+\mu)\binom{N+1}{3}+(\lambda r+\mu s+c)\binom{N}{2}}$$
$$\times \prod_{d=1}^{N} \frac{1}{P_{\mu(d+s)^m+\lambda(d+r)^k+c}} \prod_{t=1}^{d-1} \frac{P_{\lambda(d+r)^k-\lambda(t+r)^k} P_{\mu(d+s)^m-\mu(t+s)^m}}{P_{\lambda(d+r)^k+\mu(t+s)^m+c} P_{\mu(d+s)^m+\lambda(t+r)^k+c}}.$$

Similarly, in order to obtain related results for the matrix $\left[\frac{1}{L_{\lambda(i+r)^k+\mu(j+s)^m+c}}\right]_{1\le i,j}$ it is sufficient to choose $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$ in Theorem 10.

More specifically, when $q \to 1$ the entries of the matrix $\mathcal{A}$ takes the form

$$\lim_{q\to 1} \mathcal{A}_{ij} = (-1)^{-\frac{1}{2}\left[\lambda(i+r)^k+\mu(j+s)^m+c-1\right]} \frac{1}{\lambda(i+r)^k+\mu(j+s)^m+c}.$$

Since the sign function is separable with regards to the variables i and j, by using some algebraic manipulations and Theorem 9, one could obtain the results for the matrix

$$\hat{A}_{ij} = \frac{1}{\lambda(i+r)^k+\mu(j+s)^m+c},$$

which is a nonlinear generalization of the Hilbert matrix. For example, the LU-decomposition of the matrix $\hat{A}$ is obtained as

$$L_{ij} = \frac{\prod_{t=1}^{j}\left[\lambda(j+r)^k+\mu(t+s)^m+c\right]\prod_{t=1}^{j-1}\left[\lambda(i+r)^k-\lambda(t+r)^k\right]}{\prod_{t=1}^{j}\left[\lambda(i+r)^k+\mu(t+s)^m+c\right]\prod_{t=1}^{j-1}\left[\lambda(j+r)^k-\lambda(t+r)^k\right]}$$

and

$$U_{ij}=\frac{\prod_{t=1}^{i-1}\left[\lambda\left(t+r\right)^{k}-\lambda\left(i+r\right)^{k}\right]\prod_{t=1}^{i-1}\left[\mu\left(t+s\right)^{m}-\mu\left(j+s\right)^{m}\right]}{\prod_{t=1}^{i-1}\left[\lambda\left(i+r\right)^{k}+\mu\left(t+s\right)^{m}+c\right]\prod_{t=1}^{i}\left[\mu\left(j+s\right)^{m}+\lambda\left(t+r\right)^{k}+c\right]}.$$

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