Analytical Solution of the Frictional Contact Problem of a Semi-circular Punch Sliding Over a Homogeneous Orthotropic Half-plane

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ABSTRACT

An analytical solution to the frictional sliding contact problem for homogeneous orthotropic materials indented by a semi-circular punch is developed. The principal axes of orthotropy are assumed to be parallel and perpendicular to the contact. Coulomb friction assumption is used to model the friction between the punch and the orthotropic medium. The mixed boundary value problem is reduced into a Fredholm integral equation of the second kind by using Fourier transform technique. The singular integral equation is solved analytically using Jacobi Polynomials for the unknown surface contact stresses. Numerical results show the effect of the orthotropic material parameters, coefficient of friction on the contact stress distribution and load vs. contact length behavior.

Keywords: Contact mechanics, Friction, Orthotropic materials, Singular integral equation, Semi-circular punch.

INTRODUCTION

Contact mechanics problems in isotropic materials gained a great deal of interest and commonly investigated throughout the twentieth century. Orthotropic materials have been utilized both in structural design and engineering applications such as ceramic matrix composites [1]. These materials gained popularity in the last two decades and mainly projected to be used in the aerospace industry as fiber metal laminates in the structure of aircrafts and in the components of gas turbine engines [2]. For example, Tyrannohex is a high strength ceramic material containing properties of other orthotropic materials and it is utilized in the gas turbine components [3].

The studies in the theory of contact mechanics dates back to Lord Kelvin [4] who solved the problem of a force applied at a point in an isotropic infinite medium using Green's functions [5]. Then Lamé [6] further improved Lord Kelvin 's solution with superimposed stresses in a spherical container. Boussinesq [7], provided the solution of a normal force applied to the boundary of an isotropic semi-infinite solid using Green's functions and Kelvin's method. Almost at the same time Hertz [8] solved the problem involving contact between two elastic bodies with curved surfaces and postulated his famous assumptions about contact mechanics. Cerruti [9]

inquired on a problem of a force applied tangentially at the plane boundary of a semi-infinite solid also by using Kelvin's solution. In Soutwells' solution, [10] a spherical cavity in an unlimited solid under simple tension was given. Then, Mindlin [11] derived the Green's functions for the half-space by adding a supplementary part of the solution to the Kelvin's infinite space functions.

The literature on contact mechanics, especially with isotropic material assumption has been reviewed by many researchers (see for example Barber and Ciavarella, [12]). Muskhelishvili, England and Johnson [13,14,15] displayed details of the theoretical and numerical methods developed in contact mechanics. Contact problems are mixed boundary value problems due to the boundary conditions given in terms of the displacements and stresses at the same time. The formulation of these problems usually ends up with the singular integral equations (see for example Erdogan [16,17]).

In a contact problem, material selection plays a fundamental role since material properties have crucial effects on the contact stresses. Although, most of the materials contain some local heterogeneity and faults because of their manufacturing techniques, they are usually modeled as isotropic materials. Contact mec-

hanics of anisotropic materials have also been analyzed in the literature. Stroh [18,19] and Lekhnitskii [20] reported solutions using transform methods for a concentrated point force in an infinite body or on the surface of a half-space for anisotropic materials. Sveklo, [21] used integral transformation to the stress equilibrium equations and he also used the Cauchy integral for describing the boundary stress condition to solve contact problem of anisotropic material. Also, Willis [22], proposed a solution method of contact mechanics of anisotropic materials by using Fourier transform. Sveklo's method for indentation of the orthotropic half-space was analyzed by Shi et al. [23]. Kahya et.al. investigated frictionless contact problem between two orthotropic elastic layers by solving the singular integral equations [24]. Batra and Jiang's provided the parametric analysis of a punch problem for a linear elastic anisotropic layer bonded to a rigid substrate by using Stroh formalism [25]. Bagault et. al. [26] developed a semi-analytical method for the contact problem of anisotropic materials by utilizing Boussinesq and Cerruti solutions. Ashrafi et. al. [27] discussed an analytical and computational solution of the contact problem of a semi-infinite orthotropic material indented by a rigid spherical punch where a numerical analysis was presented using a finite element model. Dong et. al. [28] provided various expressions for the stresses and displacements of orthotropic materials indented by two collinear punches with flat or cylindrical profile. In addition, frictionless contact problems on arbitrarily multilayered piezoelectric half-planes modeled as orthotropic medium and solved using matrix formulation [29,30]. Recently, Zhou and Lee [31] also modeled piezoelectric half space as an orthotropic medium. They conducted a parametric analysis of two-dimensional frictionless sliding contact problem by means of the Galilean transformation [31] and they further studied a frictional contact of anisotropic piezoelectric materials indented by several stamp profiles [32].

Normally, nine independent material parameters are needed to define stress-strain behavior of an orthotropic material. Krenk [33] redefined these parameters so that the number of elastic parameters decrease to four for plane strain and generalized stress conditions. Cinar and Erdogan [34] and Ozturk and Erdogan [35,36] applied this approach to the mixed-mode crack problems in an inhomogeneous orthotropic medium.

Recently, Guler [37] developed a solution method for the sliding frictional contact problem for an orthotropic semi-infinite half space indented by a flat and a circular punch by combining Krenk's parameters and the method that he used to solve isotropic half space problems indented by various types of punch profiles [38-40]. Then, Kucuksucu et al. [41] postulated wedge-shaped indenter problem of orthotropic materials by using the same method.

The primary aim of the present study is to look into the effect of the material parameters of the contact stress distributions at the surface of the isotropic half plane indented by a rigid semi-circular punch. The problem is reduced to a Fredholm integral equation of the second type which is solved using of Jacobi Polynomials. Relationships between the applied load versus the contact length and stress intensity factors at the sharp end of the punch are also found.

Formulation of the problem

Consider the contact problem described in Fig. 1 where a rigid semi-circular punch is under sliding contact with a semi-infinite homogeneous orthotropic medium. The sliding contact is defined between $x_1 = 0$ to $x_1 = b$ at the surface of the orthotropic medium ($x_2 = 0$) where (x_1, x_2), are the principal axes of orthotropy which are parallel and perpendicular to the boundary [42,43]. It is assumed that the coefficient of static friction is constant within the contact area. P and Q are the resultant normal and shear forces, respectively, and they are proportional $Q = \eta P$, according to the Coulomb's law.

In usual notation, u_i and σ_{ij} $(i, j = 1, 2)$ specify the displacement and stress components, and E_{ii}, G_{ij} and v_{ij} $(i, j = 1, 2, 3)$ specify engineering elastic parameters. Orthotropic constitutive equations are composed of 9 elastic constants (3 Young's moduli, E_{11}, E_{22}, E_{33}, 3 shear moduli, G_{12}, G_{13}, G_{23} and 3 Poison's ratios, v_{12}, v_{13}, v_{23}). To simplify the solution, engineering parameters are replaced by four independent material parameters, namely effective stiffness parameter E, the effective Poisson's ratio v, the shear parameter κ, and stiffness ratio δ, defined by [33].

$$E = \sqrt{E_{11} E_{22}}, \quad v = \sqrt{v_{12} v_{21}}, \quad \delta^4 = \frac{E_{11}}{E_{22}} = \frac{v_{12}}{v_{21}}, \quad \kappa = \frac{E}{2G_{12}} - v,$$

(1a-d)

for generalized plane stress conditions and

$$E = \sqrt{\frac{E_{11} E_{22}}{(1 - v_{13} v_{31})(1 - v_{23} v_{32})}}, \quad v = \sqrt{\frac{(v_{12} + v_{13} v_{32})(v_{21} + v_{23} v_{31})}{(1 - v_{13} v_{31})(1 - v_{23} v_{32})}},$$

(2a,b)

$$\delta^4 = \frac{E_{11}}{E_{22}} \frac{1 - v_{23} v_{32}}{1 - v_{13} v_{31}}, \quad \kappa = \frac{E}{2G_{12}} - v,$$

(2c,d)

for plane strain conditions. In addition, we scale the independent and dependent variables by using stiffness or scaling ratio as

$$x = \frac{x_1}{\sqrt{\delta}}, \quad y = x_2\sqrt{\delta}, \quad u(x,y) = \sqrt{\delta}u_1(x_1,x_2), \quad v(x,y) = \frac{1}{\sqrt{\delta}}u_2(x_1,x_2),$$

$$\text{(3a-d)}$$

$$\sigma_{xx}(x,y) = \sigma_{11}(x_1,x_2)/\delta, \quad \sigma_{yy}(x,y) = \delta\sigma_{22}(x_1,x_2), \quad \sigma_{xy}(x,y) = \sigma_{12}(x_1,x_2).$$
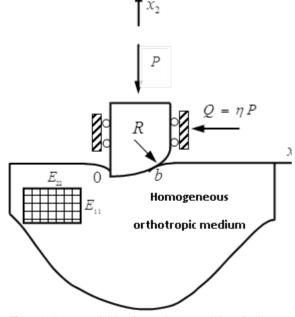
$$\text{(3e-g)}$$

In this study, the spatial variation of Poisson's ratio is assumed to be negligible, so it is taken as constant [40]. Note that the special case of $\delta = \kappa = 1$ corresponds to an isotropic material. Also, in a homogeneous orthotropic medium the range of κ can be defined as $-1 < \kappa < \infty$ and it can be shown that for $\kappa \le -1$ the elasticity problem has no applicable solution [35,36,44].

Integral equation of the problem

The singular integral equation of the sliding contact problem can be written as [37,43],

$$-\omega_1\sigma_{xy}(x,0) + \frac{1}{\pi}\int_0^{b/\sqrt{\delta}}\frac{\sigma_{yy}(t,0)}{t-x}dt = \lambda_1 E_0 f(x), \qquad 0 < x < \frac{b}{\sqrt{\delta}},$$

$$\text{(4a)}$$

$$\omega_2\sigma_{yy}(x,0) + \frac{1}{\pi}\int_0^{b/\sqrt{\delta}}\frac{\sigma_{xy}(t,0)}{t-x}dt = \lambda_2 E_0 g(x), \qquad 0 < x < \frac{b}{\sqrt{\delta}},$$

$$\text{(4b)}$$

Figure 1. Geometry of sliding frictional contact problem of orthotropic medium indented by the semi-circular punch.

where

$$f(x) = \frac{\partial}{\partial x}v(x,0), \qquad g(x) = \frac{\partial}{\partial x}u(x,0)$$

$$\text{(5a,b)}$$

$$\lambda_1 = \frac{\Delta_0}{(1-v^2)(r_2 r_4 r_7 - r_1 r_3 r_8)}, \qquad \omega_1 = \frac{2(\kappa+v)(r_1 r_3 r_6 - r_2 r_4 r_5)}{(1-v^2)(r_2 r_4 r_7 - r_1 r_3 r_8)}, \qquad \text{(6a,b)}$$

$$\lambda_2 = \frac{\Delta_0}{2(\kappa+v)(r_5-r_6)}, \qquad \omega_2 = \frac{(1-v^2)}{2(\kappa+v)}\frac{(r_8-r_7)}{(r_5-r_6)}. \qquad \text{(7a,b)}$$

In the physical domain (x_1, x_2), the integral equation (4) becomes

$$-\omega_1\tau(x_1) + \frac{\delta}{\pi}\int_0^b\frac{\sigma(t_1)}{t_1-x_1}dt_1 = \lambda_1 E_0 f(x_1), \qquad 0 < x_1 < b, \qquad \text{(8a)}$$

$$\omega_2\delta\sigma(x_1) + \frac{1}{\pi}\int_0^b\frac{\tau(t_1)}{t_1-x_1}dt_1 = \lambda_2 E_0 g(x_1), \qquad 0 < x_1 < b, \qquad \text{(8b)}$$

where

$$f(x_1) = \frac{\partial}{\partial x_1}u_2(x_1,0), \qquad g(x_1) = \delta\frac{\partial}{\partial x_1}u_1(x_1,0). \qquad \text{(9a,b)}$$

Eq. (8) constitute a pair of integral equations in terms of the unknown contact stresses σ and τ. In the contact region, we have

$$\sigma_{22}(x_1,0) = \sigma(x_1) = -p(x_1), \qquad 0 < x_1 < b, \qquad \text{(10a)}$$

$$\sigma_{12}(x_1,0) = \tau(x_1) = -\eta p(x_1), \qquad 0 < x_1 < b, \qquad \text{(10b)}$$

where the contact pressure, $p(x_1)$, $0 < x_1 < b$, is only unknown quantity. The relation between the applied load and the contact length, b can be found by applying equilibrium condition [46]. Thus, using Eq. (10), Eq. (8) become:

$$\omega_1\eta p(x_1) - \frac{\delta}{\pi}\int_0^b\frac{p(t_1)}{t_1-x_1}dt_1 = \lambda_1 E_0 f(x_1), \qquad 0 < x_1 < b, \qquad \text{(11a)}$$

$$-\omega_2\delta p(x_1) - \frac{\eta}{\pi}\int_0^b\frac{p(t_1)}{t_1-x_1}dt_1 = \lambda_2 E_0 g(x_1), \qquad 0 < x_1 < b. \qquad \text{(11b)}$$

and contact pressure must satisfy the following equilibrium equation:

$$\int_0^b p(t_1)dt_1 = P,$$

$$\text{(12)}$$

where P is the resultant compressive force. The amplitude of the applied load may be given in terms of either the load P or stamp displacement in the x_2 axis.

In order to solve the integral equation, the limits of integration must be normalized. Now setting:

$$x_1 = x_1^* R, \qquad t_1 = t_1^* R, \qquad b = b^* R, \qquad p(t_1) = p^*(t_1^*), \qquad 0 < x_1^*, t_1^* < b^*.$$
(13)

The integral equation (11a) and the equilibrium equation (12) can be written as:

$$A p^*\left(x_1^*\right) + \frac{B}{\pi} \int_0^{b^*} \frac{p^*(t_1^*)}{t_1^* - x_1^*} dt_1^* = \lambda_1 E_0 x_1^*$$
(14)

$$\int_0^{b^*} p^*(t_1^*) dt_1^* = \frac{P}{R}$$
(15)

where

$$A = \omega_1 \eta, \qquad B = -\delta.$$
(16a-b)

The integration limit is normalized from $(0, b^*)$ to $(-1, 1)$ by the following change of variables:

$$t_1^* = \frac{b^*}{2}(s+1), \qquad x_1^* = \frac{b^*}{2}(r+1), \qquad p^*(t_1^*) = \lambda_1 E_0 \frac{b^*}{2} \phi(s), \qquad -1 < r, s < 1.$$
(17a-c)

Since the stamp profile is given as $u_2(x_1, 0) = -v_0 + \frac{x_1^2}{2R}$, the function, $f(x_1)$ becomes

$$f(x_1) = \frac{\partial}{\partial x_1} u_2(x_1, 0) = \frac{x_1}{R}.$$
(18)

The integral equation (14) can then be expressed in a normalized form by using Eqs. (17) as

$$A \phi(r) + \frac{B}{\pi} \int_{-1}^1 \frac{\phi(s)}{s-r} ds = r + 1.$$
(19)

On the solution of integral equations

For an accurate and efficient solution of the integral equation the corresponding weight function $w(s)$ needs to be determined. By defining the complex potential [13,45,46]:

$$\Phi(z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\phi(s)}{s-z} ds.$$
(20)

From Muskhelishvili [13] and by using the complex function theory, the dominant part of the integral equation can be written as

$$A \phi(r) + \frac{B}{\pi} \int_{-1}^1 \frac{\phi(s)}{s-r} ds = r + 1.$$
(21)

The index of the integral equation for the semi-circular punch is defined by:

$$\chi = -(\alpha + \beta) = -(N_0 + M_0) = 0,$$
(22)

where $N_0, M_0 = -1, 0, 1$ are arbitrary integers and can be determined from the physics of the problem. Since the semi circular stamp has a sharp corner at $x_1 = 0$ and a smooth contact at $x_1 = b$, from the physics of the problem, we must require that α be positive and β be negative. α and β is found to be

$$\omega_1 \eta > 0: \qquad \alpha = \frac{\theta}{\pi}, \qquad \beta = -\frac{\theta}{\pi},$$
$$\omega_1 \eta = 0: \qquad \alpha = 0.5, \qquad \beta = -0.5,$$
$$\omega_1 \eta < 0: \qquad \alpha = 1 - \frac{\theta}{\pi}, \qquad \beta = \frac{\theta}{\pi} - 1,$$
(23a-d)

$$\theta = \arctan\left|\frac{\delta}{\omega_1 \eta}\right| > 0, \qquad 0 < \theta < \frac{\pi}{2}.$$

Now, one can assume a solution in terms of Jacobi Polynomials as:

$$\phi(s) = \sum_{n=0}^{\infty} c_n w(s) P_n^{(\alpha,\beta)}(s), \quad w(s) = (1-s)^\alpha (1+s)^\beta, \qquad -1 < s < 1,$$
(24)

where c_n, $(n = 0, 1, \ldots)$ are undetermined constants and $P_n^{(\alpha,\beta)}(s)$ are Jacobi polynomials. Substituting Eq. (24) into Eq. (21) results in

$$\sum_0^\infty c_n \left[A w(r) P_n^{(\alpha,\beta)}(r) + \frac{B}{\pi} \int_{-1}^1 \frac{w(s) P_n^{(\alpha,\beta)}(s) ds}{s-r} \right] = r + 1. \quad (25)$$

Using the following property of Jacobi polynomials:

$$AP_n^{(\alpha,\beta)}(r)w(r) + \frac{B}{\pi}\int_{-1}^1 \frac{P_n^{(\alpha,\beta)}(s)w(s)}{s-r}ds = -2^{-\chi}\frac{B}{\sin\pi\alpha}P_{n-\chi}^{(-\alpha,-\beta)}(r),$$
$$-1 < r < 1, \qquad \Re(\alpha) > 1, \qquad \Re(\beta) > 1, \qquad \Re(\alpha) \neq (0,1,\ldots,)$$
$$(26)$$

Eq. (25) can be expressed as

$$\sum_0^n c_n\left[\frac{\delta}{\sin\pi\alpha}P_n^{(-\alpha,-\beta)}(r)\right] = r+1, \qquad -1 < r < 1.$$
$$(27)$$

In this problem, after the application of a given load, one end of the contact length (i.e., b^*) is unknown. However, for a given value of the contact length (b^*) Eq. (27) gives $n+1$ equations for $n+1$ the unknowns. Expanding right hand side of Eq. (27) into a series of Jacobi polynomials $P_n^{(-\alpha,-\beta)}$ and observing that, we find:

$$r+1 = P_1^{(-\alpha,-\beta)}(r) + (1+\alpha)P_0^{(-\alpha,-\beta)}(r)$$
$$(28)$$

where

$$P_1^{(-\alpha,-\beta)}(r) = -\alpha + r, \qquad P_0^{(-\alpha,-\beta)}(r) = 1$$
$$(29a,b)$$

Therefore Eq. (27) can be written as:

$$\frac{\delta}{\sin\pi\alpha}\sum_0^n c_n P_n^{(-\alpha,-\beta)}(r) = P_1^{(-\alpha,-\beta)}(r) + (1+\alpha)P_0^{(-\alpha,-\beta)}(r),$$
$$(30)$$

Comparing right hand side and left hand side of Eq. (30), we have only two non-zero coefficients:

$$c_0 = \frac{(1+\alpha)\sin\pi\alpha}{\delta}, \qquad c_1 = \frac{\sin\pi\alpha}{\delta}.$$
$$(31a,b)$$

Therefore, the solution becomes;

$$\phi(s) = w(s)\sum_{n=0}^1 c_n P_n^{(\alpha,\beta)}(s) = w(s)\left[c_0 + c_1(\alpha+s)\right],$$
$$= w(s)\frac{\sin\pi\alpha}{\delta}[1+2\alpha+s].$$
$$(32)$$

Using Eq. (15) the equilibrium equation (17c) may be expressed as:

$$\int_{-1}^1 \phi(s)ds = \frac{4}{\lambda_1 E_0 b^{*2}}\frac{P}{R}.$$
$$(33)$$

Orthogonality condition of Jacobi Polynomials can be written as:

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(t)P_j^{(\alpha,\beta)}(t)w(t)dt = \begin{cases} 0 & n \neq j \\ \theta_j^{(\alpha,\beta)} & n = j \end{cases} \quad j = 0,1,2,\ldots$$
$$(34)$$

where

$$\theta_0^{(\alpha,\beta)} = \int_{-1}^1 w(t)dt = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$
$$(35)$$

$$\theta_j^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{(2j+\alpha+\beta+1)j!\Gamma(j+\alpha+\beta+1)}, \quad j = 1,2,\ldots$$
$$(36)$$

Using the orthogonality condition of the Jacobi Polynomials, the relation between applied load P and the contact length b can be found from Eq. (33) as:

$$c_0\theta_0 = \frac{4}{\lambda_1 E_0 b^{*2}}\frac{P}{R}.$$
$$(37)$$

θ_0 can be given as:

$$\theta_0^{(\alpha,\beta)} = \frac{2\pi\alpha}{\sin\pi\alpha}.$$
$$(38)$$

The load versus contact length relation may be obtained by substituting c_0 from Eq. (31a) and θ_0 from Eq. (38) into Eq. (37)

$$P^* = \frac{P}{E_0 R} = \frac{(1+\alpha)\pi\alpha\lambda_1}{2\delta}b^{*2}.$$
$$(39)$$

Then the contact pressure distribution $p^*(t_1^*)$ becomes:

$$p^*(t_1^*) = \lambda_1 E_0 \frac{b^*}{2}\phi(t_1^*),$$
$$= \lambda_1 E_0 \frac{b^*}{2}\left(\frac{b^*-t_1^*}{t_1^*}\right)^\alpha \sum_{n=0}^1 c_n P_n^{(\alpha,\beta)}\left(\frac{2t_1^*}{b^*}-1\right),$$
$$= \lambda_1 E_0 b^*\left(\frac{b^*-t_1^*}{t_1^*}\right)^\alpha \frac{\sin\pi\alpha}{\delta}\left[\alpha+\frac{t_1^*}{b^*}\right].$$
$$(40)$$

Using Eq. (10) and Eq. (13) the non-dimensional pressure distribution pressure becomes:

$$\frac{\sigma_{22}(x_1^*,0)}{E_0} = -\lambda_1 b^* \left(\frac{b^*-x_1^*}{x_1^*}\right)^{\alpha} \frac{\sin \pi \alpha}{\delta}\left[\alpha + \frac{x_1^*}{b^*}\right]. \quad (41)$$

The stress component $\sigma_{11}(x_1^*,0)$ can be found by using

$$\sigma_{11}\left(x_1^*,0\right) = \begin{cases} C\sigma_{22}\left(x_1^*,0\right) + \dfrac{D}{\pi}\displaystyle\int_0^{b^*}\dfrac{\sigma_{22}\left(t_1^*,0\right)}{t_1^*-x_1^*}dt_1^*, & 0 < x_1^* < b^*, \\[3mm] \dfrac{D}{\pi}\displaystyle\int_0^{b^*}\dfrac{\sigma_{22}\left(t_1^*,0\right)}{t_1^*-x_1^*}dt_1^*, & x_1^* \notin \left[0,b^*\right], \end{cases}$$

(42)

where

$$C = \left(\frac{\omega_2}{\lambda_2}+\nu\right)\delta^2, \quad D = \frac{\eta\delta}{\lambda_2}. \quad (43)$$

Therefore

$$\sigma_{11}(x_1^*,0) = -q\left(x_1^*\right) = -\lambda_1 E_0 \frac{b^*}{2}\psi\left(x_1^*\right) \quad (44)$$

$$\psi(r,0) = \begin{cases} Cw(r)c_0 P_0^{(\alpha,\beta)}\left(r\right) + Cw(r)c_1 P_1^{(\alpha,\beta)}\left(r\right) \\[2mm] +\dfrac{D}{\pi}\left\{c_0 L_0 + c_1 L_1\right\}, & -1 < r < 1, \\[4mm] \dfrac{D}{\pi}\left\{c_0 L_0 + c_1 L_1\right\}, & |r| > 1. \end{cases}$$

(45)

where

$$L_0\left(r\right) = \int_{-1}^{1}\frac{w(r)}{s-r}dr = \frac{\pi}{\sin \pi\alpha}\begin{cases}(-r+1)^{\alpha}(-r-1)^{\beta}-1, & -\infty < r < -1, \\ (1-r)^{\alpha}(1+r)^{\beta}\cos \pi\alpha -1, & -1 < r < 1, \\ (r-1)^{\alpha}(r+1)^{\beta}-1, & 1 < r < \infty, \end{cases}$$

(46)

$$L_1\left(r\right) = P_1^{(\alpha,\beta)}(r)L_0\left(r\right) + \frac{2\pi\alpha}{\sin \pi\alpha}. \quad (47)$$

Mode I stress intensity factors at the ends of the stamp for a homogeneous medium can be defined as:

$$k_p(0) = \lim_{x_1\to 0}x_1^{\alpha}p(x_1)$$

$$= \lambda_1 E_0 b^* b^{\alpha}\sum_{n=0}^{1}c_n P_n^{(\alpha,\beta)}\left(-1\right) \quad (48a)$$

$$= \lambda_1 E_0 b^* b^{\alpha}\alpha\frac{\sin \pi\alpha}{\delta}$$

Defining the non-dimensional stress intensity factors as

$$k_p^*(0) = \frac{k_p(0)}{E_0 b^{\alpha}} \quad (48b)$$

$$= \lambda_1 b^*\sum_{n=0}^{1}c_n P_n^{(\alpha,\beta)}\left(-1\right)$$

$$= \lambda_1 b^*\alpha\frac{\sin \pi\alpha}{\delta}$$

Stress intensity factor in terms of the in-plane stress component can be defined as

$$k_q(0^+) = \lim_{x_1\to 0^+}x_1^{\alpha}q(x_1)$$

$$= \lambda_1 E_0 b^* b^{\alpha}\alpha\left\{C\frac{\sin \pi\alpha}{\delta}+D\frac{\cos \pi\alpha}{\delta}\right\} \quad (49a)$$

In non-dimensional form Eq. (49a) can be expressed as

$$k_q^*(0^+) = \frac{k_q(0^+)}{Eb^{\alpha}}$$

$$= \lambda_1 b^*\alpha\left\{C\frac{\sin \pi\alpha}{\delta}+D\frac{\cos \pi\alpha}{\delta}\right\} \quad (49b)$$

$$k_q(0^-) = \lim_{x_1\to 0^-}\left(-x_1\right)^{\alpha}q(x_1) \quad (50a)$$

$$= E_0 b^*\left(b\right)^{\alpha}\frac{\lambda_1\eta}{\lambda_2}\alpha$$

Similarly, in non-dimensional form Eq. (50a) can be expressed as:

$$k_q^*(0^-) = \frac{k_q(0^-)}{\left(b\right)^{\alpha}E_0}$$

$$= \frac{\lambda_1\eta}{\lambda_2}b^*\alpha \quad (50b)$$

RESULTS AND DISCUSSION

Contact problem described in Fig. 1 is solved analytically to obtain results for the contact stresses and in-plane

stress distributions beneath semi-circular punch profile under various restrictions. In the results, the contact stresses are normalized by E_0. Results are given for the following range of parameters ($-0.1 \leq \kappa \leq 5$, $0.2 \leq \delta^4 \leq 5$, $\nu = 3/7$ and $0 \leq \eta \leq 0.9$). There are certain limitations on the material parameters of orthotropic materials. These restrictions require that $\kappa + \nu > 0$, (see Eq.(1) and (2), $0 < \nu < 1$ and $\kappa > -1$.

Fig. 2-4 illustrate the contact pressure, $\sigma_{22}(x_1, 0)$ under semi-circular punch. Note that the contact pressure is bounded and zero at the smooth end of semi-circular punch $(x_1 = b)$. However, at the leading or another words sharp end, the contact stress is singular. In-plane stresses, $\sigma_{11}(x_1, 0)$ are bounded and discontinuous at the leading

edge $(x_1 = 0)$. In the distribution of $\sigma_{11}(x_1, 0)$ as $(x_1 \rightarrow b)$ near leading edge needle-like spikes distribution is observed.

This case, obviously results in crack nucleation and as a result component total service life may be reduced because of contact fracture [47]. It is interesting that neither the stiffness ratio, δ, nor the shear parameter, κ, has effect on the distribution of in-plane stress, $\sigma_{11}(x_1, 0)$, at the leading edge $(x_1 \rightarrow b)$ because of the formulation as

$$\frac{\sigma_{11}(b,0)}{E_0} = -\lambda_1 \frac{b^*}{2} \psi(1) = b^* \eta \qquad (51)$$

Fig. 6a shows the dependence of various material parameters δ and the κ on the powers of stress singularities, α and β for fixed value of the coefficient of friction,
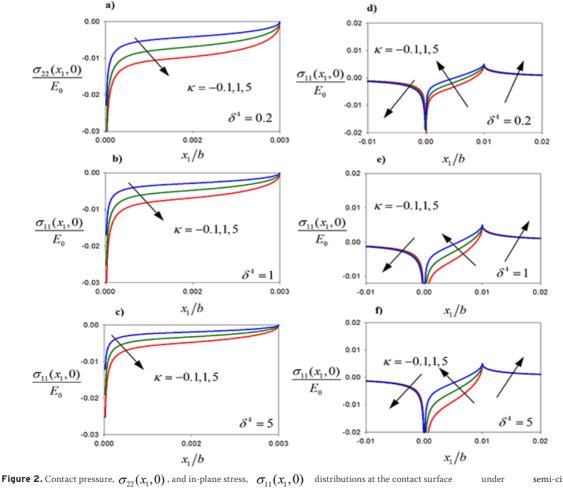

Figure 2. Contact pressure, $\sigma_{22}(x_1, 0)$, and in-plane stress, $\sigma_{11}(x_1, 0)$ distributions at the contact surface under semi-circular punch for various values of the parameters $\kappa = \dfrac{E}{2G_{12}} - \nu$ with $\eta = 0.5$, $\nu = 3/7$, $b/R = 0.01$, $\delta^4 = \frac{E_{11}}{E_{22}} = \frac{\nu_{12}}{\nu_{21}}$ where E and ν are given in equations (1) and (2) a) $\sigma_{22}(x_1, 0)$ for $\delta^4 = 0.2$;b) $\sigma_{22}(x_1, 0)$ for $\delta^4 = 1$; c) $\sigma_{22}(x_1, 0)$ for $\delta^4 = 5$; d) $\sigma_{11}(x_1, 0)$ for $\delta^4 = 0.2$; e) $\sigma_{11}(x_1, 0)$ for $\delta^4 = 1$; f) $\sigma_{11}(x_1, 0)$ for $\delta^4 = 5$.
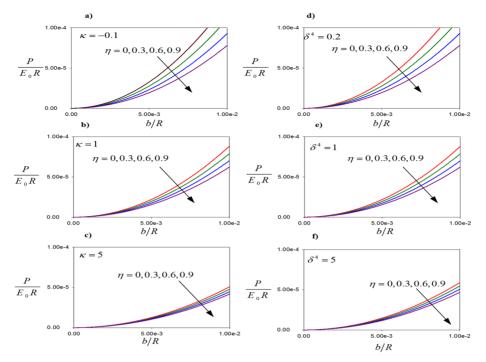
Figure 3. Contact pressure, $\sigma_{22}(x_1,0)$, and in-plane stress, $\sigma_{11}(x_1,0)$ distributions at the contact surface under semi-circular punch for various values of the parameters $\kappa = \dfrac{E}{2G_{12}} - \nu$ with $\eta = 0.5$, $\nu = 3/7$, $b/R = 0.01$, $\delta^4 = \dfrac{E_{11}}{E_{22}} = \dfrac{\nu_{12}}{\nu_{21}}$ where E and ν are given in equations (1) and (2) a) $\sigma_{22}(x_1,0)$ for $\kappa = -0.1$; b) $\sigma_{22}(x_1,0)$ for $\kappa = 1$; c) $\sigma_{22}(x_1,0)$ for $\kappa = 5$; d) $\sigma_{11}(x_1,0)$ for $\kappa = -0.1$; e) $\sigma_{11}(x_1,0)$ for $\kappa = 1$; f) $\sigma_{11}(x_1,0)$ for $\kappa = 5$.
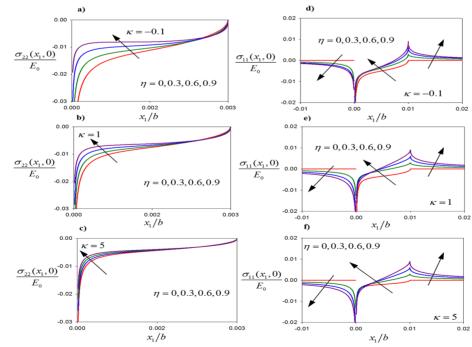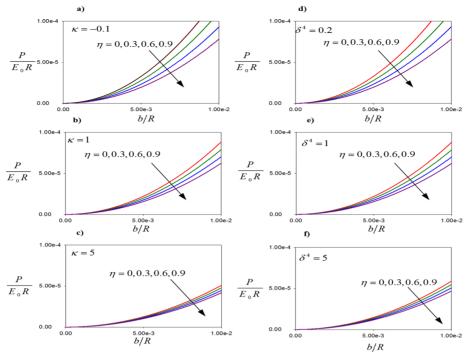

Figure 4. Contact pressure, $\sigma_{22}(x_1,0)$ and in-plane stress $\sigma_{11}(x_1,0)$ distributions at the contact surface under semi-circular punch for various values of the friction coefficients parameters η, with $\kappa = \dfrac{E}{2G_{12}} - \nu$, $\nu = 3/7$, $b/R = 0.01$, $\delta^4 = \dfrac{E_{11}}{E_{22}} = \dfrac{\nu_{12}}{\nu_{21}} = 0.2$ where E and ν are given equations (1) and (2) a) $\sigma_{22}(x_1,0)$ for $\kappa = -0$; $\kappa = 1$; c) $\sigma_{22}(x_1,0)$ for $\kappa = 5$; d) $\sigma_{11}(x_1,0)$ for $\kappa = -0.1$; e) $\sigma_{11}(x_1,0)$ for $\kappa = 1$; f) $\sigma_{11}(x_1,0)$ for $\kappa = 5$.

Figure 5. The load, $\dfrac{P}{E_0 R}$ and the contact length b, an orthotropic homogeneous medium under semi-circular punch for various values of the friction coefficients η, with $\kappa = \dfrac{E}{2G_{12}} - \nu$, $\nu = 3/7$, $b/R = 0.01$, $\delta^4 = \dfrac{E_{11}}{E_{22}} = \dfrac{\nu_{12}}{\nu_{21}}$ where E and ν are given equations (1) and (2) a) $\kappa = -0.1$, $\delta^4 = 0.2$;b) $\kappa = 1$, $\delta^4 = 0.2$ c) $\kappa = 5$, $\delta^4 = 0.2$; $\kappa = -0.1$;f) $\delta^4 = 5$, $\kappa = -0.1$.

$\eta = 0.5$, and effective Poisson's ratio, $\nu = 3/7$. As the shear parameter, κ, increases, $|\alpha|$ increases for fixed values of the stiffness ratio parameter, δ. Note that, for $\kappa > 3$ the change of the δ has no effect on the curves. Fig. 6b depicts the dependence of κ and δ on the powers of stress singularities, α and β for fixed value of the coefficient of friction, $\eta = 0.5$, and effective Poisson's ratio, $\nu = 3/7$. As the stiffness ratio parameter, δ, increases, $|\alpha|$ increases for fixed values of the shear parameter, κ. Note that, for $\delta > 3$ the curves do not sensitive to the change of the κ.

Table 1 shows some examples of the stress intensity factors obtained for a semi-circular stamp. The values of stress intensity factors increase both shear parameter and stiffness ratio decreases.

Table 1.The normalized stress intensity factors for a homogeneous orthotropic medium under contact stresses for the semi-circular punch, $\nu = 3/7$.
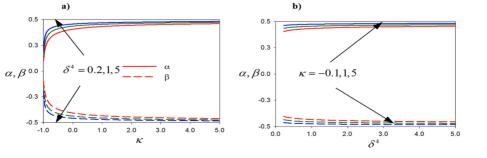

Figure 6. Strength of stress singularity at $x_1 = b$, α and $x_1 = 0$, β with $\eta = 0.5$, , $\nu = 3/7$ for various values of a) , $\delta^4 = \dfrac{E_{11}}{E_{22}} = \dfrac{\nu_{12}}{\nu_{21}}$ b) $\kappa = \dfrac{E}{2G_{12}} - \nu$ where E and ν are given in equations (2) and (3) for semi-circular punch where $\chi = -(\alpha + \beta) = 0$.

Stiffness ratio	$\eta = 0.5$			$\eta = 0.9$		
	$\kappa = -0.1$	$\kappa = 1$	$\kappa = 5$	$\kappa = -0.1$	$\kappa = 1$	$\kappa = 5$
$\delta^4 = 0.2$	0.00426	0.00316	0.00197	0.003232	0.00267	0.00181
$\delta^4 = 1$	0.00315	0.00225	0.00136	0.002668	0.00203	0.00129
$\delta^4 = 5$	0.00224	0.00156	0.00092	0.002028	0.00146	0.00090

Stiffness ratio	$\eta = 0.5$			$\eta = 0.9$		
	$\kappa = -0.1$	$\kappa = 1$	$\kappa = 5$	$\kappa = -0.1$	$\kappa = 1$	$\kappa = 5$
$\delta^4 = 0.2$	0.00251	0.00186	0.00116	0.002941	0.002433	0.001650
$\delta^4 = 1$	0.00360	0.00257	0.00155	0.003904	0.002974	0.001892
$\delta^4 = 5$	0.00534	0.00371	0.00221	0.005474	0.003964	0.002429

Stiffness ratio	$\eta = 0.5$			$\eta = 0.9$		
	$\kappa = -0.1$	$\kappa = 1$	$\kappa = 5$	$\kappa = -0.1$	$\kappa = 1$	$\kappa = 5$
$\delta^4 = 0.2$	0.002009	0.00216	0.00230	0.00300	0.00344	0.00387
$\delta^4 = 1$	0.002166	0.00227	0.00236	0.00345	0.00377	0.00407
$\delta^4 = 5$	0.002274	0.00234	0.00241	0.00378	0.00401	0.00421

CONCLUSION

In this paper, an analytical solution to the plane contact problem is given on orthotropic homogeneous medium is intended by a sliding rigid semi-circular stamp. The given problem is reduced to a second kind singular integral equation, which is solved using of Jacobi Polynomials. The effect of orthotropic material parameters and friction coefficient on the contact stress are presented. The following conclusions can be drawn from the results found in this study:

• In sliding contact problems orthotropic homogeneous materials the weight functions $w(x)$ describing the asymptotic behavior of the contact stresses are dependent, as in the isotropic homogeneous materials, on the coefficient of friction η and the surface value of the Poisson's ratio ν (or the shear parameter κ) only, and are independent of all other material constants and length parameters.

• In-plane stress tensile spike occurs on the surface at the trailing end of the contact region. The magnitude of the tensile spike increases with the increasing coefficient of friction, η stiffness ratio, δ and shear parameter κ.

• In all cases the resultant force P increases with increasing contact area in a parabolic manner.

• The shear parameter κ, and the stiffness ratio δ. do not affect the length of the contact zone.

• The Poisson ratio ν has only negligible influence on the $\sigma_{22}(x_1,0)$ contact pressure distribution for $\kappa \leq -0.1$

• Results have relevance to surface crack initiation and propagation in load transfer components.

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