FORMULÆ FOR TWO WEIGHTED BINOMIAL IDENTITIES WITH THE FALLING FACTORIALS

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Abstract. In this paper, we will give closed formulæ for weighted and alternating weighted binomial sums with the generalized Fibonacci and Lucas numbers including both falling factorials and powers of indices. Furthermore we will derive closed formulæ for weighted binomial sums including odd powers of the generalized Fibonacci and Lucas numbers.

1. Introduction

For $n > 1$, define the generalized Fibonacci and Lucas sequences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} - U_{n-2} \text{ and } V_n = pV_{n-1} - V_{n-2},$$

with $U_0 = 0$, $U_1 = 1$, and $V_0 = 2$, $V_1 = p$, respectively. The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2 - 4}\right)/2$.

From [2], recall that for $k \geq 0$ and $n > 1$,

$$U_{kn} = V_k U_{k(n-1)} - U_{k(n-2)} \quad \text{and} \quad V_{kn} = V_k V_{k(n-1)} - V_{k(n-2)}.$$

As generalizations of the results of [9], Prodinger [8] derived a general formula for the sum

$$\sum_{i=1}^{n} F_{2i+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0, 1\}$, as well as for the corresponding sums for Lucas numbers.

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After this Kılıç et. al [4] derived general formulæ for the alternating sums

$$\sum_{i=1}^{n}(-1)^i F_{2i+\delta}^{2m+\varepsilon} \text{ and } \sum_{i=1}^{n}(-1)^i L_{2i+\delta}^{2m+\varepsilon}.$$

Khan and Kwong [7] studied the sums

$$\sum_{i=0}^{n}\binom{n}{i} i^m U_i \text{ and } \sum_{i=0}^{n}\binom{n}{i}(-1)^i i^m U_i.$$

In [5], the authors computed alternating binomial sums

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i f(n,i,k,t) \text{ and } \sum_{i=0}^{n}\binom{n}{i} g(n,i,k,t),$$

where $f(n,i,k,t)$ and $g(n,i,k,t)$ are certain products of generalized Fibonacci and Lucas numbers.

Kılıç et. al [3] computed the sums

$$\sum_{i=0}^{n}\binom{n}{i} i^s U_{ki}^{2s+\varepsilon}, \quad \sum_{i=0}^{n}\binom{n}{i} i^s V_{ki}^{2s+\varepsilon},$$

as well as their alternating analogues for positive integers k and s where ε is defined as before.

By inspiring from [3, 5], the authors [6] derived formulæ for the binomial sums

$$\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}}(-1)^i f(n,i,k,t),$$

where $f(n,i,k,t)$ is defined as before and m is a nonnegative integer and $x^{\underline{m}}$ stands for the falling factorial defined by $x^{\underline{m}} = x(x-1)\dots(x-m+1)$.

In this paper, we compute the weighted binomial sums

$$\sum_{i=0}^{n}\binom{n}{i} i^{s+\underline{m}} g(i,k) \quad \text{and} \quad \sum_{i=0}^{n}\binom{n}{i}(-1)^i i^{s+\underline{m}} g(i,k),$$

where $g(i,k)$ is either U_{ki}^{2s+1} or $V_{ki}^{2s+1}$for $k, m > 0$.

2. The Main results

Before our main results, we give some auxiliary results. For $n \geq 2$, define the sequences $\{X_{kn}\}, \{Y_{kn}\}, \{W_{kn}\}$ and $\{Z_{kn}\}$ as

$$X_0 = 0,\ X_k = U_k,\ X_{kn} = (V_k + 2)\left(X_{k(n-1)} - X_{k(n-2)}\right),$$
$$Y_0 = 0,\ Y_k = U_k,\ Y_{kn} = (V_k - 2)\left(Y_{k(n-1)} + Y_{k(n-2)}\right),$$
$$W_0 = 2,\ W_k = V_k + 2,\ W_{kn} = (V_k + 2)\left(W_{k(n-1)} - W_{k(n-2)}\right),$$
$$Z_0 = 2,\ Z_k = V_k - 2,\ Z_{kn} = (V_k - 2)\left(Z_{k(n-1)} + Z_{k(n-2)}\right).$$

The Binet formulæ are

$$\begin{aligned} X_{kn} &= \frac{\left(1+\alpha^k\right)^n-\left(1+\beta^k\right)^n}{\alpha-\beta}, \quad Y_{kn}=\frac{\left(\alpha^k-1\right)^n-\left(\beta^k-1\right)^n}{\alpha-\beta}, \\ W_{kn} &= \left(1+\alpha^k\right)^n+\left(1+\beta^k\right)^n \text{ and } Z_{kn}=\left(\alpha^k-1\right)^n+\left(\beta^k-1\right)^n, \end{aligned}$$

where $\alpha^k, \beta^k = \left(V_k \pm \sqrt{V_k^2-4}\right)/2$.

From (see Eq. (1.118) on page 36, [1]), we recall the following lemma:

Lemma 1 ([1]). *For nonnegative integers n and m,*

$$\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} a^i = a^m n^{\underline{m}}(1+a)^{n-m}\left[a \neq -1 \text{ and } m \neq n\right].$$

We need the following result.

Theorem 1. *For nonnegative integers n and m,*

$$\begin{aligned} \sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} U_{ki} &= \frac{n^{\underline{m}}}{\left(2+V_k\right)^m} X_{k(n+m)}, \\ \sum_{i=0}^{n}\binom{n}{i} i^{1+\underline{m}} U_{ki} &= \frac{n^{\underline{m}}}{\left(2+V_k\right)^m}\left(nX_{k(n+m)}+(m-n)X_{k(n+m-1)}\right), \\ \sum_{i=0}^{n}\binom{n}{i}(-1)^i\, i^{\underline{m}} U_{ki} &= (-1)^{n+m}\frac{n^{\underline{m}}}{\left(2-V_k\right)^m} Y_{k(n+m)}, \\ \sum_{i=0}^{n}\binom{n}{i}(-1)^i\, i^{1+\underline{m}} U_{ki} &= \frac{n^{\underline{m}}(-1)^{n+m-1}}{\left(2-V_k\right)^m}\left((m-n)\,Y_{k(n+m-1)}-nY_{k(n+m)}\right). \end{aligned}$$

Proof. Consider

$$\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} U_{ki} = \frac{1}{\alpha-\beta}\left[\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}}\alpha^{ki}-\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}}\beta^{ki}\right],$$

which, by Lemma 1, equals

$$\frac{n^{\underline{m}}}{\alpha-\beta}\left[\frac{\left(1+\alpha^k\right)^n}{\left(1+\beta^k\right)^m}-\frac{\left(1+\beta^k\right)^n}{\left(1+\alpha^k\right)^m}\right] = \frac{n^{\underline{m}}}{\left(2+V_k\right)^m} X_{k(n+m)},$$

as claimed. One can easily obtain the rest of claimed identities. □

Similar to the proof of Theorem 1, we have the following result without proof.

Theorem 2. *For nonnegative integers n and m,*

$$\sum_{i=0}^{n}\binom{n}{i} i^{\underline{m}} V_{ki} = \frac{n^{\underline{m}}}{\left(2+V_k\right)^m} W_{k(n+m)},$$

$$\sum_{i=0}^{n}\binom{n}{i}i^{1+\underline{m}}V_{ki}=\frac{n^{\underline{m}}}{(2+V_k)^m}\left[(m-n)W_{k(n+m-1)}+nW_{k(n+m)}\right],$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\,i^{\underline{m}}V_{ki}=(-1)^{n+m}\frac{n^{\underline{m}}}{(2-V_k)^m}Z_{k(n+m)},$$
$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\,i^{1+\underline{m}}V_{ki}=\frac{n^{\underline{m}}(-1)^{n+m-1}}{(2-V_k)^m}\left[(m-n)Z_{k(n+m-1)}-nZ_{k(n+m)}\right].$$

In order to generalize Theorems 1 and 2, we will define two new operators. For $n\geq 1$, define the operators D_U and Δ_U on $X_{k(n+m)}$ and $Y_{k(n+m)}$ as follows

$$D_U\left(X_{k(n+m)}\right)=nX_{k(n+m)}+(m-n)X_{k(n+m-1)}, \tag{2.1}$$
$$\Delta_U\left(Y_{k(n+m)}\right)=nY_{k(n+m)}-(m-n)Y_{k(n+m-1)}. \tag{2.2}$$

For example, from Theorem 1 and (2.1), we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}i^{2+\underline{m}}U_{ki}&=D_U\left[\sum_{i=0}^{n}\binom{n}{i}i^{\underline{m}+1}U_{ki}\right]\\&=D_U\left[\frac{n^{\underline{m}}}{(2+V_k)^m}\left(nX_{k(n+m)}+(m-n)X_{k(n+m-1)}\right)\right]\\&=\frac{n^{\underline{m}}}{(2+V_k)^m}\left[n^2X_{k(n+m)}+(m-n)(2n-1)X_{k(n+m-1)}\right.\\&\left.+(m-n)(m-n+1)X_{k(n+m-2)}\right].\end{aligned}$$

From the discussion above, if $\sum\limits_{i=0}^{n}\binom{n}{i}i^{s-1+\underline{m}}U_{ki}$ is of the form $\frac{n^{\underline{m}}}{(2+V_k)^m}\sum\limits_{t\geq 0}a_tX_t$, then

$$\sum_{i=0}^{n}\binom{n}{i}i^{s+\underline{m}}U_{ki}=\frac{n^{\underline{m}}}{(2+V_k)^m}D_U\left[\sum_{t\geq 0}a_tX_t\right].$$

Hence the coefficients a_t can be computed iteratively. Iterative process is summarized in the theorem:

Theorem 3. *The polynomials $a_{s,r}(m,n)$ satisfy the recurrence*

$$a_{s,r}(m,n)=(n-r)a_{s-1,r}(m,n)+(m-n+r-1)a_{s-1,r-1}(m,n),\ \ s\geq 1,$$

where the initial value $a_{0,0}(m,n)=1$ and if $r<0$ or $r>s$, $a_{s,r}(m,n)=0$. For any integers $m,s\geq 0$,

$$i)\quad \sum_{i=0}^{n}\binom{n}{i}i^{s+\underline{m}}U_{ki}=\frac{n^{\underline{m}}}{(2+V_k)^m}\sum_{r=0}^{s}a_{s,r}(m,n)X_{k(n+m-r)}, \tag{2.3}$$

$$ii) \quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\, i^{s+\underline{m}}U_{ki}=\frac{n^{\underline{m}}\,(-1)^{n+m}}{(2-V_k)^{m}}\sum_{r=0}^{s}(-1)^{r}\, a_{s,r}(m,n)Y_{k(n+m-r)}. \tag{2.4}$$

Proof. i) Recall that

$$\sum_{i=0}^{n}\binom{n}{i}i^{s+\underline{m}}U_{ki}=D_U\left(\sum_{i=0}^{n}\binom{n}{i}i^{s-1+\underline{m}}U_{ki}\right).$$

Thus by (2.1), we have

$$\begin{aligned}
&\sum_{r=0}^{s}a_{s,r}(m,n)X_{k(n+m-r)}=D_U\left[\sum_{r=0}^{s-1}a_{s-1,r}(m,n)X_{k(n+m-r)}\right]\\
&=\sum_{r=0}^{s-1}a_{s-1,r}(m,n)\left((n-r)X_{k(n+m-r)}+(m-n+r)X_{k(n+m-r-1)}\right)\\
&=na_{s-1,0}(m,n)X_{k(n+m)}+\sum_{r=1}^{s-1}(n-r)a_{s-1,r}(m,n)X_{k(n+m-r)}\\
&+\sum_{r=1}^{s-1}(m-n+r-1)a_{s-1,r-1}(m,n)X_{k(n+m-r)}\\
&+(m-n+s-1)a_{s-1,s-1}(m,n)X_{k(n+m-s)}\\
&=na_{s-1,0}(m,n)X_{k(n+m)}+(m-n+s-1)a_{s-1,s-1}(m,n)X_{k(n+m-s)}\\
&+\sum_{r=1}^{s-1}\left((n-r)a_{s-1,r}(m,n)+(m-n+r-1)a_{s-1,r-1}(m,n)\right)X_{k(n+m-r)}.
\end{aligned}$$

Since $a_{s-1,r}(m,n)=0$ if $r<0$ or $r>s-1$, we write

$$\begin{aligned}
&\sum_{r=0}^{s}a_{s,r}(m,n)X_{k(n+m-r)}\\
&=\sum_{r=0}^{s}\left[(n-r)a_{s-1,r}(m,n)+(m-n+r-1)a_{s-1,r-1}(m,n)\right]X_{k(n+m-r)}.
\end{aligned}$$

The recurrence of $a_{s,r}(m,n)$ follows by comparing coefficients.

ii) Observing

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\, i^{s+\underline{m}}U_{ki}=\Delta_U\left[\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\, i^{s-1+\underline{m}}U_{ki}\right],$$

the proof follows similar to the first claim. □

For $n \geq 1$, define the operators D_U and Δ_U on $W_{k(n+m)}$ and $Z_{k(n+m)}$ as

$$D_V\left(W_{k(n+m)}\right) = nW_{k(n+m)} + (m-n)W_{k(n+m-1)},$$
$$\Delta_V\left(Z_{k(n+m)}\right) = nZ_{k(n+m)} - (m-n)Z_{k(n+m-1)}.$$

Theorem 4. *For $m, s \geq 0$,*

$$\sum_{i=0}^{n}\binom{n}{i} i^{s+\underline{m}} V_{ki} = \frac{n^{\underline{m}}}{(2+V_k)^m}\sum_{r=0}^{s} a_{s,r}(m,n) W_{k(n+m-r)}, \tag{2.5}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i i^{s+\underline{m}} V_{ki} = \frac{n^{\underline{m}}(-1)^{n+m}}{(2-V_k)^m}\sum_{r=0}^{s}(-1)^r a_{s,r}(m,n) Z_{k(n+m-r)}. \tag{2.6}$$

Proof. The proof is similar to the proof of Theorem 3. □

3. Additional Sums Formulæ including odd powers of the Generalized Fibonacci and Lucas numbers

In this section, we will derive much more general case of the results of Theorems 3 and 4 by taking odd powers of the generalized Fibonacci and Lucas numbers. Before this, we need to recall some facts.

From [10], for reals m and n, recall that

$$(m+n)^k = \sum_{i=0}^{(k-1)/2}\binom{k}{i}(mn)^i(m^{k-2i}+n^{k-2i}) \quad \text{if } k \text{ is odd},$$

and

$$(m-n)^k = \sum_{i=0}^{(k-1)/2}\binom{k}{i}(-1)^i (mn)^i(m^{k-2i}-n^{k-2i}) \quad \text{if } k \text{ is odd}. \tag{3.1}$$

Now we are ready to give our first claim:

Theorem 5. *For $k, s > 0$,*

$$\sum_{i=0}^{n}\binom{n}{i} i^{s+\underline{m}} U_{ki}^{2s+1} = \frac{n^{\underline{m}} U_k^{2s}}{(V_k^2-4)^s}\sum_{j=0}^{s}(-1)^j\binom{2s+1}{j}\frac{1}{(2+V_{k(2s-2j+1)})^m}$$
$$\times \sum_{r=0}^{s} a_{s,r}(m,n) X_{k(2s-2j+1)(n+m-r)}.$$

Proof. For $k > 0$, by the Binet formula of $\{U_n\}$ and (3.1), we have

$$\sum_{i=0}^{n}\binom{n}{i} i^{s+\underline{m}} U_{ki}^{2s+1} = \sum_{i=0}^{n}\binom{n}{i} i^{s+\underline{m}}\left(\frac{\alpha^{ki}-\beta^{ki}}{\alpha-\beta}\right)^{2s+1}$$

$$= \frac{1}{(\alpha-\beta)^{2s+1}} \sum_{i=0}^{n} \binom{n}{i} i^{\underline{s+m}}$$

$$\times \sum_{j=0}^{s} \binom{2s+1}{j} (-1)^j \left(\alpha^{ki(2s-2j+1)} - \beta^{ki(2s-2j+1)}\right)$$

$$= \frac{1}{(p^2-4)^s} \sum_{j=0}^{s} \binom{2s+1}{j} (-1)^j \sum_{i=0}^{n} \binom{n}{i} i^{\underline{s+m}} U_{ki(2s-2j+1)},$$

which, by taking $k(2s+1-2j)$ replace of k in (2.3), equals

$$\frac{n^{\underline{m}} U_k^{2s}}{(V_k^2-4)^s} \sum_{j=0}^{s} \binom{2s+1}{j} \frac{(-1)^j}{\left(2+V_{k(2s-2j+1)}\right)^m} \sum_{r=0}^{s} a_{s,r}(m,n) X_{k(2s-2j+1)(n+m-r)},$$

as claimed. □

Theorem 6. *For $k, s > 0$,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i i^{\underline{s+m}} U_{ki}^{2s+1}$$

$$= (-1)^{n+m} \frac{n^{\underline{m}} U_k^{2s}}{(V_k^2-4)^s} \sum_{j=0}^{s} (-1)^j \binom{2s+1}{j} \frac{1}{(2-V_{k(2s-2j+1)})^m}$$

$$\times \sum_{r=0}^{s} (-1)^r a_{s,r}(m,n) Y_{k(2s-2j+1)(n+m-r)}.$$

Proof. For $k > 0$, consider

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i i^{\underline{s+m}} U_{ki}^{2s+1} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i i^{\underline{s+m}} \left(\frac{\alpha^{ki}-\beta^{ki}}{\alpha-\beta}\right)^{2s+1},$$

which, by (3.1), equals

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i i^{\underline{s+m}} \left[\sum_{j=0}^{s} \binom{2s+1}{j} (-1)^j \left(\frac{\alpha^{ki(2s-2j+1)} - \beta^{ki(2s-2j+1)}}{(\alpha-\beta)^{2s+1}}\right)\right]$$

$$= \frac{1}{(p^2-4)^s} \sum_{j=0}^{s} \binom{2s+1}{j} (-1)^j \sum_{i=0}^{n} \binom{n}{i} (-1)^i i^{\underline{s+m}} U_{ki(2s-2j+1)}.$$

By taking $k(2s+1-2j)$ instead of k in (2.4), the claimed result follows. □

Using (2.5) and (2.6), and, by following the proof of Theorem 5, we have the following result without proof.

Theorem 7. *For $k, s > 0$,*

$$\sum_{i=0}^{n}\binom{n}{i}i^{s+\underline{m}}V_{ki}^{2s+1}=n^{\underline{m}}\sum_{j=0}^{s}\binom{2s+1}{j}\frac{1}{\left(2+V_{k(2s-2j+1)}\right)^{m}}$$
$$\times\sum_{r=0}^{s}a_{s,r}(m,n)W_{k(2s-2j+1)(n+m-r)}.$$

Theorem 8. *For $k,s>0$,*

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\,i^{s+\underline{m}}V_{ki}^{2s+1}=(-1)^{n+m}\,n^{\underline{m}}$$
$$\times\sum_{j=0}^{s}\binom{2s+1}{j}\frac{1}{\left(2-V_{k(2s-2j+1)}\right)^{m}}\sum_{r=0}^{s}(-1)^{r}\,a_{s,r}(m,n)Z_{k(2s-2j+1)(n+m-r)}.$$

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