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NEW RECIPROCAL SUMS INVOLVING FINITE PRODUCTS OF SECOND ORDER RECURSIONS

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Abstract. In this paper, we present new kinds of reciprocal sums of finite products of general second order linear recurrences. In order to evaluate explicitly them by q-calculus, first we convert them into their q-notation and then use the methods of partial fraction decomposition and creative telescoping.

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1. INTRODUCTION

For $n > 1$, define the second order linear recurrences $\{U_n\}$ and $\{V_n\}$ with

$$U_n = pU_{n-1} + U_{n-2} \text{ and } V_n = pV_{n-1} + V_{n-2},$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively.

If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number). If $p = 2$, then $U_n = P_n$ (nth Pell number) and $V_n = Q_n$ (nth Pell-Lucas number).

The Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^{n-1}(1-q^n)}{(1-q)} \text{ and } V_n = \alpha^n + \beta^n = \alpha^n(1+q^n),$$

where $\alpha, \beta = (p \pm \sqrt{p^2+4})/2$, $q = \beta/\alpha$ and $\mathbf{i} = \sqrt{-1}$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\ldots(1-xq^{n-1})$. When $x = q$, we denote $(q;q)_n$ by $(q)_n$.

Many authors [1–11] have studied both finite or infinite and alternating or non-alternating reciprocal sums including terms of certain integer sequences. More recently, Frontczak [3] evaluated various reciprocal sums of the Fibonacci numbers. For example, he showed that for $m, n \geq 1$

$$\sum_{i=1}^{N} (-1)^{m(i+1)} \frac{F_{mi+n\mp 1}}{F_{m(i-1)+n} F_{mi+n} F_{m(i+1)+n}} = \frac{F_{m\mp 1}}{F_n F_m F_{2m}}$$

$$\times\left(\frac{F_{m(N+1)}}{F_{m(N+1)+n}}+\frac{F_{mN}}{F_{mN+n}}-\frac{F_m}{F_{m+n}}\right)-(-1)^m\frac{F_{mN}}{F_m^2F_{m+n}F_{m(N+1)+n}}.$$

Kılıç and Prodinger [5] consider some classes of reciprocal sums of general Fibonacci numbers, which were computed in closed form in an earlier work and then they evaluate the same sums by a different approach: First they convert them into their q-forms and then explicitly evaluate q-versions of the sums by using partial fraction decomposition method and creative telescoping idea.

In this paper, we shall consider new kinds of reciprocal sums including finite products of the general Fibonacci and Lucas numbers. We shall summarize what we will present in this paper below.

- We consider three kinds of *alternating* reciprocal sums including finite products of the general Fibonacci and Lucas numbers. All of them have an integer parameter to increase or decrease the indices of the general Fibonacci or Lucas factors in both numerator and denominator in the sums. Two of these sums are of the forms

$$\sum_{k=0}^{n}(-1)^k\frac{U_{k-d}}{U_{k+d}U_{k+d+1}U_{k+d+2}} \text{ and } \sum_{k=0}^{n}(-1)^k\frac{V_{k+d+1}}{U_{k+d}U_{k+d+1}U_{k+d+2}}.$$

  The other kind sums has an additional parameter m which determines the number of the general Fibonacci or Lucas factors in the numerator or denominator. It will be of the form

$$\sum_{k=0}^{n}(-1)^k\frac{U_{k+c}U_{k+c+1}\dots U_{k+c+m-1}}{X_{k+d}X_{k+d+1}\dots X_{k+d+m+1}},$$

  where X_n is either U_n or V_n.
- Before all, we convert all the claimed results into their q-forms. After this, we will use partial fraction decomposition (pfd) method and creative telescoping idea to prove the q-version of the claimed results.
- By the q-versions of three kinds of reciprocal sums, we present general cases of the original three kinds of sums with an additional integer parameter by taking a special choosing of q. By the way, we could able to derive *non-alternating* reciprocal sums where the indices of the general Fibonacci or Lucas numbers are in the arithmetic progressions.
- Finally we shall give some applications of our results to certain alternating and non-alternating reciprocal sums with finite products of the Pell or Pell-Lucas numbers.

2. The main results

Now we are going to present our main results. We start with our first main result.

Theorem 1. *For $n, m \geq 0$ and $c \in \{-m+1, \ldots, -1, 0, 1\}$,*

$$\sum_{k=0}^{n}(-1)^k \frac{U_{k+c}U_{k+c+1}\ldots U_{k+c+m-1}}{X_{k+d}X_{k+d+1}\ldots X_{k+d+m+1}} = (-1)^{c+1}\frac{U_{n+c}U_{n+c+1}\ldots U_{n+c+m}}{U_{m+1}X_{d-c+1}\left[X_{n+d+1}X_{n+d+2}\ldots X_{n+d+m+1}\right]},$$

where X_n is either U_n or V_n. Note that when $X_n = V_n$, we assume that d is any integer. When $X_n = U_n$, we assume that $d \geq 1$.

Proof. Let $X_n = U_n$. First we convert the LHS of the claim into its q-notation as shown

$$\sum_{k=0}^{n}(-1)^k \frac{\prod_{t=0}^{m-1} U_{k+t+c}}{\prod_{t=0}^{m+1} U_{k+t+d}} = \alpha^{mc-(m+2)d-2m+1}(1-q)^2 \sum_{k=0}^{n} \frac{q^k \prod_{t=0}^{m-1}\left(1-q^{k+c+t}\right)}{\prod_{t=0}^{m+1}(1-q^{k+d+t})}.$$

Second we convert the RHS of the claim into its q-notation as shown

$$(-1)^{c+1}\frac{\prod_{t=0}^{m} U_{n+t+c}}{U_{m+1}U_{d-c+1}\prod_{t=1}^{m+1} U_{n+t+d}} = (-1)^{c+1}\alpha^{(m+2)c-(m+2)d-2m-1}\frac{(1-q)^2 \prod_{t=0}^{m}\left(1-q^{n+c+t}\right)}{(1-q^{m+1})(1-q^{d-c+1})\prod_{t=1}^{m+1}(1-q^{n+d+t})}.$$

After some simplifications, the q-version of the claimed result is

$$\sum_{k=0}^{n} \frac{q^k \prod_{t=0}^{m-1}\left(1-q^{k+c+t}\right)}{\prod_{t=0}^{m+1}(1-q^{k+d+t})} = \frac{q^{1-c}\prod_{t=0}^{m}\left(1-q^{n+c+t}\right)}{(1-q^{m+1})(1-q^{d-c+1})\prod_{t=1}^{m+1}(1-q^{n+d+t})}$$

or, in terms of the q-Pochhammer notation,

$$\sum_{k=0}^{n} \frac{q^k \left(q^{k+c};q\right)_m}{\left(q^{k+d};q\right)_{m+2}} = \frac{q^{1-c}\left(q^{n+c};q\right)_{m+1}}{(1-q^{m+1})(1-q^{d-c+1})\left(q^{n+d+1};q\right)_{m+1}}.$$

Define

$$S_n := \sum_{k=0}^{n} \frac{q^k \left(q^{k+c};q\right)_m}{\left(q^{k+d};q\right)_{m+2}}.$$

Denote the summand term of S_n by T_k, that is,

$$T_k = \frac{q^k \left(q^{k+c};q\right)_m}{\left(q^{k+d};q\right)_{m+2}}.$$

The partial fraction decomposition of T_k reads as

$$\frac{q^k \left(q^{k+c};q\right)_m}{\left(q^{k+d};q\right)_{m+2}} = \sum_{t=1}^{m+2} \frac{q^{-d-t+1}\left(q^{c-d-t+1};q\right)_m}{\left(1-q^{k+d+t-1}\right)\prod\limits_{\substack{i=1\\ i\neq t}}^{m+2}\left(1-q^{i-t}\right)}.$$

From [5], we could obtain the following identity comes from creative telescoping idea:

$$\sum_{k=0}^{n}\left(\frac{1}{1-q^{k+d+m}}-\frac{1}{1-q^{k+d+t}}\right) = \sum_{k=0}^{m-t-1}\left(\frac{1}{1-q^{k+d+n+t+1}}-\frac{1}{1-q^{k+d+t}}\right). \tag{2.1}$$

In that case, by using the equality (2.1), we write

$$\begin{aligned}
S_n &= \sum_{t=1}^{m+1}\left(\frac{1}{1-q^{d+t-1}}-\frac{1}{1-q^{d+n+t}}\right)\sum_{r=1}^{t}\frac{q^{-d-r+1}\left(q^{c-d-r+1};q\right)_m}{\left(q^{1-r};q\right)_{r-1}(q)_{m-r+2}} \\
&= (-1)^m \frac{1}{1-q^{m+1}}\sum_{t=1}^{m+1}\left(\frac{1}{1-q^{d+t-1}}-\frac{1}{1-q^{d+n+t}}\right) \\
&\quad \times q^{(m-t+1)c-(m-t+2)d+t(t-1)+\frac{m(m-2t+1)}{2}}\frac{\left(q^{c-d-2};q\right)_{3-t}\left(q^{d-c};q\right)_{t-m+1}}{(q)_{t-1}(q)_{m-t+1}} \\
&= (-1)^m \frac{1-q^{n+1}}{1-q^{m+1}}\sum_{t=1}^{m+1}\frac{q^{t(-c+d-m+t)}}{(1-q^{d+t-1})(1-q^{d+n+t})} \\
&\quad \times q^{c-1-d+\frac{1}{2}m(m+1+2c-2d)}\frac{\left(q^{c-d-2};q\right)_{3-t}\left(q^{d-c};q\right)_{t-m+1}}{(q)_{t-1}(q)_{m-t+1}} \\
&= \frac{1-q^{n+1}}{(1-q^{c-d-1})(1-q^{m+1})}
\end{aligned}$$

$$\times\sum_{t=1}^{m+1}(-1)^{t+1}\frac{q^{\binom{t+1}{2}-1}\left(q^{c-d-1};q\right)_{2-t}\left(q^{c-d};q\right)_{m-t+1}}{(1-q^{d+t-1})(1-q^{d+n+t})\,(q)_{t-1}\,(q)_{m-t+1}}$$

$$=\frac{1-q^{n+1}}{(1-q^{c-d-1})(1-q^{m+1})}$$

$$\times\sum_{t=0}^{m}(-1)^{t}\frac{q^{\binom{t+2}{2}-1}\left(q^{c-d-1};q\right)_{1-t}\left(q^{c-d};q\right)_{m-t}}{(1-q^{d+t})(1-q^{d+n+t+1})\,(q)_{t}\,(q)_{m-t}}$$

$$=\frac{1-q^{n+1}}{(1-q^{m+1})(1-q^{c-d-1})}\sum_{t=0}^{m}(-1)^{t}\frac{q^{\binom{t}{2}}}{(q^{-t}-q^{d})(q^{-t}-q^{d+n+1})}$$

$$\times\frac{(1-q^{c-d-t-1})(1-q^{c-d-t})\dots(1-q^{c-d+m-t-2})(1-q^{c-d+m-t-1})}{(q)_{t}\,(q)_{m-t}}.$$

Now we define

$$h(z):=\frac{(1-zq^{c-d-1})\dots(1-zq^{c-d+m-1})}{(z-q^{d})(z-q^{d+n+1})(1-z)(1-zq)\dots(1-zq^{m})}.$$

The partial fraction decomposition of $h(z)$ is

$$h(z)\;=\frac{(1-q^{c-1})\dots(1-q^{c+m-1})}{(z-q^{d})(q^{d}-q^{d+n+1})(1-q^{d})(1-q^{d+1})\dots(1-q^{d+m})}$$

$$+\frac{(1-q^{c+n})\dots(1-q^{c+n+m})}{(q^{d+n+1}-q^{d})(z-q^{d+n+1})(1-q^{d+n+1})(1-q^{d+n+2})\dots(1-q^{d+n+m+1})}$$

$$+\sum_{t=0}^{m}(-1)^{t}q^{\binom{t+1}{2}}\frac{(1-q^{-t+c-d-1})\dots(1-q^{-t+c-d+m-1})}{(q^{-t}-q^{d})(q^{-t}-q^{d+n+1})\,(q)_{t}\,(q)_{m-t}\,(1-zq^{t})}.$$

If we multiply $h(z)$ by z and let $z\to\infty$, then we get

$$0=\frac{q^{-d}(1-q^{c-1})\dots(1-q^{c+m-1})}{(1-q^{n+1})(1-q^{d})(1-q^{d+1})\dots(1-q^{d+m})}$$

$$-\frac{q^{-d}(1-q^{c+n})\dots(1-q^{c+n+m})}{(1-q^{n+1})(1-q^{d+n+1})(1-q^{d+n+2})\dots(1-q^{d+n+m+1})}$$

$$+\sum_{t=0}^{m}(-1)^{t+1}q^{\binom{t}{2}}\frac{(1-q^{-t+c-d-1})\dots(1-q^{-t+c-d+m-1})}{(q^{-t}-q^{d})(q^{-t}-q^{d+n+1})\,(q)_{t}\,(q)_{m-t}}.$$

Since $c\in\{-m+1,\dots,-1,0,1\}$, we write

$$\sum_{t=0}^{m}(-1)^{t}q^{\binom{t}{2}}\frac{(1-q^{-t+c-d-1})\dots(1-q^{-t+c-d+m-1})}{(q^{-t}-q^{d})(q^{-t}-q^{d+n+1})\,(q)_{t}\,(q)_{m-t}}$$

$$= -\frac{q^{-d}(1-q^{n+c})\dots(1-q^{c+m+n})}{(1-q^{n+1})(1-q^{d+n+1})(1-q^{d+n+2})\dots(1-q^{d+m+n+1})}.$$

Taking into the constant factor, we write

$$\begin{aligned}
&\frac{1-q^{n+1}}{(1-q^{m+1})(1-q^{c-d-1})} \\
&\quad\times\sum_{t=0}^{m}(-1)^t q^{\binom{t}{2}+1}\frac{(1-zq^{c-d-1})\dots(1-zq^{c-d+m-1})}{(z-q^d)(z-q^{d+n+1})\,(q)_t\,(q)_{m-t}} \\
&= -\frac{q^{-d}(1-q^{n+c})\dots(1-q^{c+m+n})}{(1-q^{m+1})(1-q^{c-d-1})(1-q^{d+n+1})\dots(1-q^{d+m+n+1})} \\
&= -\frac{q^{-d}\left(q^{n+c};q\right)_{m+1}}{(1-q^{m+1})(1-q^{c-d-1})\left(q^{n+d+1};q\right)_{m+1}} \\
&= \frac{q^{1-c}\left(q^{n+c};q\right)_{m+1}}{(1-q^{m+1})(1-q^{d-c+1})\left(q^{n+d+1};q\right)_{m+1}},
\end{aligned}$$

which completes the proof.

Also, when $X_n = V_n$, the proof is similarly obtained. □

Now we will present some interesting corollaries of Theorem 1. When $m = 2$, $X_n = L_n$, $U_n = F_n$, $d = 3$ and $c = 0$ in Theorem 1, it gives us

$$\sum_{k=0}^{n}(-1)^k\frac{F_kF_{k+1}}{L_{k+3}L_{k+4}L_{k+5}L_{k+6}} = -\frac{F_nF_{n+1}F_{n+2}}{F_3L_4L_{n+4}L_{n+5}L_{n+6}}.$$

When $m = 3$, $X_n = U_n = P_n$, $d = 5$ and $c = 1$ in Theorem 1, we get

$$\sum_{k=0}^{n}(-1)^k\frac{P_{k+1}P_{k+2}P_{k+3}}{P_{k+5}P_{k+6}P_{k+7}P_{k+8}P_{k+9}} = \frac{P_{n+1}P_{n+2}P_{n+3}P_{n+4}}{P_4P_5P_{n+6}P_{n+7}P_{n+8}P_{n+9}}.$$

When $m = 4$, $X_n = Q_n$, $U_n = P_n$, $d = 4$ and $c = -2$ in Theorem 1, we get

$$\sum_{k=0}^{n}(-1)^k\frac{P_{k-2}P_{k-1}P_kP_{k+1}}{Q_{k+4}Q_{k+5}Q_{k+6}Q_{k+7}Q_{k+8}Q_{k+9}} = -\frac{P_{n-2}P_{n-1}P_nP_{n+1}P_{n+2}}{P_5Q_7Q_{n+5}Q_{n+6}Q_{n+7}Q_{n+8}Q_{n+9}}.$$

When $m = 5$, $X_n = U_n = F_n$, $d = 5$ and $c = -3$ in Theorem 1, we get

$$\sum_{k=0}^{n}(-1)^k\frac{F_{k-3}F_{k-2}F_{k-1}F_kF_{k+1}}{F_{k+5}F_{k+6}F_{k+7}F_{k+8}F_{k+9}F_{k+10}F_{k+11}}$$

$$= \frac{F_{n-3}F_{n-2}F_{n-1}F_nF_{n+1}F_{n+2}}{F_6F_9F_{n+6}F_{n+7}F_{n+8}F_{n+9}F_{n+10}F_{n+11}}.$$

Now we shall give our second result. The first main result stands for the finite product of the general Fibonacci or Lucas numbers. But the next two results are valid for special cases.

Theorem 2. *For $d > 0$,*

$$\sum_{k=0}^{n}(-1)^k \frac{U_{k-d}}{U_{k+d}U_{k+d+1}U_{k+d+2}} = (-1)^{d+1}\frac{U_{2n+2}}{U_2U_{n+d+1}U_{n+d+2}}. \tag{2.2}$$

Proof. First we convert the LHS of (2.2) into its q-notation as shown

$$\sum_{k=0}^{n}(-1)^k \frac{U_{k-d}}{U_{k+d}U_{k+d+1}U_{k+d+2}}$$

$$= \alpha^{-4d-1}(1-q)^2 \sum_{k=0}^{n} \frac{q^k(1-q^{k-d})}{(1-q^{k+d})(1-q^{k+d+1})(1-q^{k+d+2})}.$$

Second we convert the RHS of (2.2) into its q-notation as shown

$$(-1)^{d+1}\frac{U_{2n+2}}{U_2U_{n+d+1}U_{n+d+2}} = \alpha^{-2d-1}(1-q)^2\frac{(-1)^{d+1}(1-q^{2n+2})}{\left(1-q^2\right)(1-q^{d+n+1})(1-q^{d+n+2})}.$$

After some simplifications, q-version of the claimed result is

$$\sum_{k=0}^{n}\frac{q^k(1-q^{k-d})}{(1-q^{k+d})(1-q^{k+d+1})(1-q^{k+d+2})} = -\frac{q^{-d}(1-q^{2n+2})}{\left(1-q^2\right)(1-q^{d+n+1})(1-q^{d+n+2})}.$$

Define

$$S_n := \sum_{k=0}^{n}\frac{z(1-zq^{-d})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}.$$

Denote the summand term of S_n by $T(z)$, that is,

$$T(z) = \frac{z(1-zq^{-d})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}.$$

The partial fraction decomposition of $T(z)$ reads as

$$\frac{z(1-zq^{-d})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}$$

$$= \frac{1}{q^{3d+1}(1-q)^2(1+q)}\left(-\frac{q(1-q^{2d})}{1-zq^d} + \frac{(1+q)(1-q^{2d+1})}{1-zq^{d+1}} - \frac{1-q^{2d+2}}{1-zq^{d+2}}\right)$$

$$= \frac{1}{q^{3d+1}(1-q)^2(1+q)}\left[q(1-q^{2d})\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^d}\right)\right.$$

$$-(1+q)(1-q^{2d+1})\left(\frac{1}{1-zq^{d+2}}-\frac{1}{1-zq^{d+1}}\right)\Bigg].$$

And so

$$S_n=\frac{1}{q^{3d+1}(1-q)^2(1+q)}\left[q(1-q^{2d})\sum_{k=0}^{n}\left(\frac{1}{1-zq^{d+2}}-\frac{1}{1-zq^{d}}\right)\right. \tag{2.3}$$
$$\left.-(1+q)(1-q^{2d+1})\sum_{k=0}^{n}\left(\frac{1}{1-zq^{d+2}}-\frac{1}{1-zq^{d+1}}\right)\right].$$

If we take $m=2$, $t=0$, and, $m=2$, $t=1$ in (2.1), respectively, then we rewrite S_n given in (2.3) as

$$\frac{1}{q^{3d+1}(1-q)^2(1+q)}\left[q(1-q^{2d})\sum_{k=0}^{1}\left(\frac{1}{1-q^{k+d+1+n}}-\frac{1}{1-q^{k+d}}\right)\right.$$
$$\left.-(1+q)(1-q^{2d+1})\left(\frac{1}{1-q^{d+n+2}}-\frac{1}{1-q^{d+1}}\right)\right]$$
$$=-\frac{\left(1+q^{n+1}\right)\left(1-q^{d+1}\right)\left(1-q^{n+1}\right)}{q^{d}\left(1-q^{d+1}\right)\left(1-q^{d+n+1}\right)\left(1-q^{d+n+2}\right)(1-q)(1+q)}$$
$$=-\frac{q^{-d}\left(1-q^{2n+2}\right)}{\left(1-q^{2}\right)(1-q^{d+n+1})(1-q^{d+n+2})}.$$

Thus, we have the conclusion. □

When $U_n=F_n$ and $d=3$ in (2.2), we obtain

$$\sum_{k=0}^{n}(-1)^k\frac{F_{k-3}}{F_{k+3}F_{k+4}F_{k+5}}=\frac{F_{2n+2}}{F_{n+4}F_{n+5}}.$$

Now we going to give our third result:

Theorem 3. *For $d>0$,*

$$\sum_{k=0}^{n}(-1)^k\frac{V_{k+d+1}}{U_{k+d}U_{k+d+1}U_{k+d+2}}=\frac{U_{n+1}U_{n+2(d+1)}}{U_1U_dU_{d+1}U_{n+d+1}U_{n+d+2}}. \tag{2.4}$$

Proof. After required converting and simplifications, we find the q-version of the claimed result as follows

$$\sum_{k=0}^{n}\frac{q^k(1+q^{k+d+1})}{(1-q^{k+d})(1-q^{k+d+1})(1-q^{k+d+2})}$$
$$=\frac{(1-q^{n+1})(1-q^{n+2(d+1)})}{(1-q)\left(1-q^{d}\right)\left(1-q^{d+1}\right)(1-q^{n+d+1})(1-q^{n+d+2})}.$$

Define

$$S_n := \sum_{k=0}^{n} \frac{z(1+zq^{d+1})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}.$$

Denote the summand term of S_n by $T(z)$, that is,

$$T(z) = \frac{z(1+zq^{d+1})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})}.$$

The partial fraction decomposition of $T(z)$ reads as

$$\begin{aligned}
&\frac{z(1+zq^{d+1})}{(1-zq^d)(1-zq^{d+1})(1-zq^{d+2})} \\
&= \frac{1}{q^d(1+q)(1-q)^2}\left(\frac{1+q}{1-zq^d} - \frac{2(1+q)}{1-zq^{d+1}} + \frac{1+q}{1-zq^{d+2}}\right) \\
&= \frac{-1}{q^d(1+q)(1-q)^2}\left[(1+q)\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^d}\right)\right. \\
&\left. -2(1+q)\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^{d+1}}\right)\right].
\end{aligned}$$

And so

$$\begin{aligned}
S_n =& \frac{-1}{q^d(1+q)(1-q)^2}\left[(1+q)\sum_{k=0}^{n}\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^d}\right)\right. \\
&\left. -2(1+q)\sum_{k=0}^{n}\left(\frac{1}{1-zq^{d+2}} - \frac{1}{1-zq^{d+1}}\right)\right].
\end{aligned}$$

If we take $m=2$, $t=0$, and, $m=2$, $t=1$ in (2.1), respectively, then we rewrite the last equality as

$$\begin{aligned}
S_n =& \frac{-1}{q^d(1+q)(1-q)^2}\left[(1+q)\sum_{k=0}^{1}\left(\frac{1}{1-q^{k+d+1+n}} - \frac{1}{1-q^{k+d}}\right)\right. \\
&\left. -2(1+q)\left(\frac{1}{1-q^{d+n+2}} - \frac{1}{1-q^{d+1}}\right)\right] \\
=& \frac{(1-q^{n+1})(1-q^{n+2d+2})}{(1-q)(1-q^d)(1-q^{d+1})(1-q^{n+d+1})(1-q^{n+d+2})}.
\end{aligned}$$

Thus, we have the conclusion. □

When $U_n = F_n$, $V_n = L_n$ and $d = 5$ in (2.4), we obtain

$$\sum_{k=0}^{n}(-1)^k \frac{L_{k+6}}{F_{k+5}F_{k+6}F_{k+7}} = \frac{F_{n+1}F_{n+12}}{F_5F_6F_{n+6}F_{n+7}}.$$

3. General cases

In the previous section, we found the q-versions of the claimed results in Theorems 1-3 while proving them. Now we shall present more general cases of Theorems 1-3 by taking $q = \beta^s/\alpha^s$ for any integer s in the q-forms of them without proof, respectively.

Theorem 4. *For $n, m \geq 0$ and $c \in \{-m+1, \ldots, -1, 0, 1\}$,*

$$\sum_{k=0}^{n} (-1)^{sk} \frac{\prod_{t=0}^{m-1} U_{s(k+c+t)}}{\prod_{t=0}^{m+1} X_{s(k+d+t)}} = (-1)^{s(c+1)} \frac{\prod_{t=0}^{m} U_{s(n+t+c)}}{U_{s(m+1)} X_{s(d-c+1)} \prod_{t=1}^{m+1} X_{s(n+t+d)}}, \tag{3.1}$$

where X_n is either U_n or V_n. Note that when $X_n = V_n$, we assume that d is any integer. When $X_n = U_n$, we assume that $d \geq 1$.

In (3.1), if we take $m = 3$, $s = 2$, $U_n = P_n$, $X_n = Q_n$, $d = 4$ and $c = 1$, then we obtain

$$\sum_{k=0}^{n} \frac{P_{2k+2} P_{2k+4} P_{2k+6}}{Q_{2k+8} Q_{2k+10} Q_{2k+12} Q_{2k+14} Q_{2k+16}} = \frac{P_{2n+2} P_{2n+4} P_{2n+6} P_{2n+8}}{P_8 Q_8 Q_{2n+10} Q_{2n+12} Q_{2n+14} Q_{2n+16}}.$$

When $m = 4$, $s = -1$, $U_n = X_n = F_n$, $d = 6$ and $c = 0$ in Theorem 4, then, by $F_{-n} = (-1)^{n+1} F_n$, we obtain

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^k \frac{F_{-k} F_{-k-1} F_{-k-2} F_{-k-3}}{F_{-k-6} F_{-k-7} F_{-k-8} F_{-k-9} F_{-k-10} F_{-k-11}} \\ &= \sum_{k=0}^{n} (-1)^k \frac{F_k F_{k+1} F_{k+2} F_{k+3}}{F_{k+6} F_{k+7} F_{k+8} F_{k+9} F_{k+10} F_{k+11}} \\ &= -\frac{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}}{F_5 F_7 F_{n+7} F_{n+8} F_{n+9} F_{n+10} F_{n+11}}. \end{aligned}$$

When $m = 5$, $s = 3$, $U_n = F_n$, $X_n = L_n$, $d = 2$ and $c = -3$ in Theorem 4, then we obtain

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^k \frac{F_{3k-9} F_{3k-6} F_{3k-3} F_{3k} F_{3k+3}}{L_{3k+6} L_{3k+9} L_{3k+12} L_{3k+15} L_{3k+18} L_{3k+21} L_{3k+24}} \\ &= \frac{F_{3n-9} F_{3n-6} F_{3n-3} F_{3n} F_{3n+3} F_{3n+6}}{F_{18} L_{18} L_{3n+9} L_{3n+12} L_{3n+15} L_{3n+18} L_{3n+21} L_{3n+24}}. \end{aligned}$$

Theorem 5. *For $d > 0$,*

$$\sum_{k=0}^{n}(-1)^{sk}\frac{U_{s(k-d)}}{U_{s(k+d)}U_{s(k+d+1)}U_{s(k+d+2)}} = (-1)^{sd+1}\frac{U_{s(2n+2)}}{U_{2s}U_{s(n+d+1)}U_{s(n+d+2)}}. \tag{3.2}$$

If we take $U_n = F_n$, $d = 2$ and $s = 5$ in (3.2), then we obtain

$$\sum_{k=0}^{n}(-1)^{k}\frac{F_{5k-10}}{F_{5k+10}F_{5k+15}F_{5k+20}} = -\frac{F_{10n+10}}{F_{10}F_{5n+15}F_{5n+20}}.$$

Theorem 6. *For $d > 0$,*

$$\sum_{k=0}^{n}(-1)^{sk}\frac{V_{s(k+d+1)}}{U_{s(k+d)}U_{s(k+d+1)}U_{s(k+d+2)}} = \frac{U_{s(n+1)}U_{s(n+2d+2)}}{U_sU_{sd}U_{s(d+1)}U_{s(n+d+1)}U_{s(n+d+2)}}. \tag{3.3}$$

If we take $U_n = P_n$, $V_n = Q_n$, $d = 4$ and $s = -3$ in (3.3), then, by $P_{-n} = (-1)^{n+1}P_n$ and $Q_{-n} = (-1)^nQ_n$, we obtain

$$\sum_{k=0}^{n}(-1)^{k}\frac{Q_{-3k-15}}{P_{-3k-12}P_{-3k-15}P_{-3k-18}} = \sum_{k=0}^{n}(-1)^{k}\frac{Q_{3k+15}}{P_{3k+12}P_{3k+15}P_{3k+18}}$$
$$= \frac{P_{3n+3}P_{3n+30}}{P_3P_{12}P_{15}P_{3n+15}P_{3n+18}}.$$

4. Conclusions

In the last section, we derived more general results, where the sign functions and the indices of Fibonacci or Lucas factors are depend on the integer parameter s. By the way, one can see that these generalized reciprocal sums are alternating for odd integer s, and, non-alternating for even integer s while the original sums in Theorems 1-3 are always alternating sums.

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