ON BINOMIAL DOUBLE SUMS WITH FIBONACCI AND LUCAS NUMBERS-II

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ABSTRACT. In this paper, we compute various binomial-double-sums involving the Fibonacci numbers as well as their alternating analogous. It would be interesting that all sums we shall compute are evaluated in nice multiplication forms in terms of again the Fibonacci and Lucas numbers.

1. INTRODUCTION

Define second order linear recurrences $\{U_n, V_n\}$ as for $n > 0$

$$\begin{aligned} U_n &= pU_{n-1} + U_{n-2}, \\ V_n &= pV_{n-1} + V_{n-2}, \end{aligned}$$

where $U_0 = 0,\ U_1 = 1$, and $V_0 = 2,\ V_1 = p$, resp.

If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number). The Binet formulæ are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(1 \pm \sqrt{5}\right)/2$.

By the Binet formulæ of F_n and L_n, for later use one can see that

$$F_{-n} = (-1)^{n+1} F_n \text{ and } L_{-n} = (-1)^n L_n.$$

Much recently, Kılıç and Taşdemir [1] consider and compute various sum families of binomial sums namely *binomial-double-sums* including double sums and one binomial coefficient of the forms

$$\sum_{0 \le i,j \le n} \binom{i}{j} U_{ri+tj}, \quad \sum_{0 \le i,j \le n} \binom{i}{j} V_{ri+tj}$$

as well as their alternating analogues

$$\sum_{0 \le i,j \le n} \binom{i}{j} (-1)^i U_{ri+tj}, \quad \sum_{0 \le i,j \le n} \binom{i}{j} (-1)^i V_{ri+tj}$$

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for some integers r and t.

For example, they showed that let t and r be odd integers. For nonnegative even k,

$$\sum_{0\leq i,j\leq k}\binom{i}{j}U_{ri+2tj}=\frac{\Delta^{\frac{k}{2}}U_t^{k+1}\left[V_{(t+r)(k+1)}+\Delta U_tU_{k(t+r)}\right]-U_tV_{t+r}}{\Delta U_t^2+\Delta U_tU_{t+r}-1}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}V_{ri+2tj}$$
$$=\frac{\Delta^{\frac{k}{2}+1}U_t^{k+1}\left[U_tV_{k(t+r)}+U_{(t+r)(k+1)}\right]+\Delta U_tU_{t+r}-2}{\Delta U_t^2+\Delta U_tU_t-1}.$$

For positive odd k,

$$\sum_{0\leq i,j\leq k}\binom{i}{j}U_{ri+2tj}=\frac{\Delta^{\frac{k+1}{2}}U_t^{k+1}\left[U_tV_{k(t+r)}+U_{(t+r)(k+1)}\right]-U_tV_{t+r}}{\Delta U_t^2+\Delta U_tU_{t+r}-1}$$

and

$$\sum_{0\leq i,j\leq k}\binom{i}{j}V_{ri+2tj}$$
$$=\frac{\Delta^{\frac{k+1}{2}}U_t^{k+1}[\Delta U_tU_{k(t+r)}+V_{(t+r)(k+1)}]+\Delta U_tU_{t+r}-2}{\Delta U_t^2+\Delta U_tU_{t+r}-1}.$$

The authors of [1] also compute other kinds alternating analogues of these sums whose signs are of the forms $(-1)^j$ and $(-1)^{i+j}$. These sums are evaluated via certain linear combinations of terms U_n and V_n that are not in multiplication form. Also for earlier similar binomial sums families, we could refer to the reference list of [1].

In this paper, as a second part of *binomial-double-sums,* we present sums families including the Fibonacci numbers. But in this part, all sums we shall compute are evaluated in nice multiplication form in terms of again the Fibonacci and Lucas numbers.

2. Binomial-Double-Sums with the Fibonacci Numbers

In this section, we will present our binomial-double-sums including the Fibonacci numbers. By the Binomial theorem, first we start with recalling an auxiliary lemma from [1].

Lemma 1. *For any real numbers x and y such that $x(1+y)\neq 1$*

$$\sum_{0\leq i,j\leq k}\binom{i}{j}x^iy^j=\frac{(x+xy)^{k+1}-1}{x+xy-1}.$$

As some consequences of Lemma 1, for later use, we could note the following identities:

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^i x^i y^j = \frac{(-1)^k\left(x+xy\right)^{k+1}+1}{x+xy+1}, \quad x(1+y)\neq -1, \tag{2.1}$$

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^j x^i y^j = \frac{\left(x-xy\right)^{k+1}-1}{x-xy-1}, \quad x(1-y)\neq 1, \tag{2.2}$$

and

$$\sum_{0\le i,j\le k}\binom{i}{j}(-1)^{i+j} x^i y^j = \frac{(-1)^k\left(x-xy\right)^{k+1}+1}{x-xy+1}, \quad x(1-y)\neq -1. \tag{2.3}$$

For later use, we also present a Fibonacci-Lucas identity. As a showcase, we give a proof for the first item of the following Lemma and the others could be proven similarly.

Lemma 2. *For integers m and n,*

$$F_{2(m+n)} - F_{2m} - F_{2n} = \begin{cases} 5F_mF_nF_{m+n} & \text{if } m \text{ and } n \text{ are even,} \\ L_mL_nF_{m+n} & \text{if } m \text{ and } n \text{ are odd,} \\ L_mF_nL_{m+n} & \text{if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

Proof. By $\alpha-\beta=\sqrt{5}$ and $\alpha\beta=-1$, consider the RHS of the claim for even m and n,

$$\begin{aligned}
&5F_mF_nF_{m+n} \\
&= \frac{5(\alpha^m-\beta^m)(\alpha^n-\beta^n)(\alpha^{m+n}-\beta^{m+n})}{(\alpha-\beta)^3} \\
&= \frac{(\alpha^{m+n}+\beta^{m+n}-(\alpha\beta)^m(\beta^{n-m}+\alpha^{n-m}))(\alpha^{m+n}-\beta^{m+n})}{\alpha-\beta} \\
&= \frac{\alpha^{2m+2n}-\beta^{2m+2n}-(\alpha^{2n}-\beta^{2n}+(\alpha\beta)^n(\beta^{-m}\alpha^m-\alpha^{-m}\beta^m))}{\alpha-\beta} \\
&= \frac{\alpha^{2m+2n}-\beta^{2m+2n}-(\alpha^{2n}-\beta^{2n})-(\alpha^{2m}-\beta^{2m})}{\alpha-\beta} \\
&= F_{2m+2n}-F_{2n}-F_{2m},
\end{aligned}$$

as claimed. □

We already have the following two lemmas from [2].

Lemma 3. *For any integers m and n,*

$$\begin{aligned}
F_{n+m}-(-1)^mF_{n-m} &= F_mL_n, \\
F_{n+m}+(-1)^mF_{n-m} &= L_mF_n.
\end{aligned}$$

Lemma 4. *For any integer n,*

$$L_{2n} - 2(-1)^n = 5F_n^2 \quad \text{and} \quad L_{2n} + 2(-1)^n = L_n^2.$$

Now we shall give our first result.

Theorem 1. *For all nonnegative integer n and any integer t*

(1)

$$\sum_{0 \le i,j \le n} \binom{i}{j} F_{i+j} = \frac{1}{2} \begin{cases} F_{\frac{3n}{2}} L_{\frac{3n+4}{2}} & \text{if } n \equiv 0 (\text{mod } 4), \\ F_{\frac{3n+1}{2}} L_{\frac{3(n+1)}{2}} & \text{if } n \equiv 1 (\text{mod } 4), \\ L_{\frac{3n}{2}} F_{\frac{3n+4}{2}} & \text{if } n \equiv 2 (\text{mod } 4), \\ L_{\frac{3n+1}{2}} F_{\frac{3(n+1)}{2}} & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

(2)

$$\sum_{0 \le i,j \le n} \binom{i}{j} F_{4ti+j} = \frac{1}{L_{2t+1}} \begin{cases} F_{(2t+1)n} L_{(2t+1)(n+1)} & \text{if } n \text{ is even}, \\ L_{(2t+1)n} F_{(2t+1)(n+1)} & \text{if } n \text{ is odd}. \end{cases}$$

(3)

$$\sum_{0 \le i,j \le n} \binom{i}{j} F_{2(2t+1)i+j} = \frac{F_{2n(t+1)} F_{2(n+1)(t+1)}}{F_{2(t+1)}} \quad (\text{for } t \neq -1).$$

(4)

$$\sum_{0 \le i,j \le n} \binom{i}{j} F_j = \begin{cases} F_n L_{n+1} & \text{if } n \text{ is even}, \\ L_n F_{n+1} & \text{if } n \text{ is odd}. \end{cases}$$

(5)

$$\sum_{0 \le i,j \le n} \binom{i}{j} F_{2i-j} = \frac{1}{2} \begin{cases} F_{\frac{3n}{2}} L_{\frac{3n+4}{2}} & \text{if } n \equiv 0 (\text{mod } 4), \\ F_{\frac{3n+1}{2}} L_{\frac{3(n+1)}{2}} & \text{if } n \equiv 1 (\text{mod } 4), \\ L_{\frac{3n}{2}} F_{\frac{3n+4}{2}} & \text{if } n \equiv 2 (\text{mod } 4), \\ L_{\frac{3n+1}{2}} F_{\frac{3(n+1)}{2}} & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

(6)

$$\sum_{0 \le i,j \le n} \binom{i}{j} F_{(4t+1)i-j} = \frac{1}{L_{2t+1}} \begin{cases} F_{(2t+1)n} L_{(2t+1)(n+1)} & \text{if } n \text{ is even}, \\ L_{(2t+1)n} F_{(2t+1)(n+1)} & \text{if } n \text{ is odd}. \end{cases}$$

(7)

$$\sum_{0 \le i,j \le n} \binom{i}{j} F_{(4t+3)i-j} = \frac{F_{2n(t+1)} F_{2(n+1)(t+1)}}{F_{2(t+1)}} \quad (\text{for } t \neq -1).$$

Proof. We only prove the first and third identities. We choose the first item of the first identity. If $n \equiv 0 (\mathrm{mod}\ 4)$, then assume that $n = 4k$ for $k \in \mathbb{Z}$. Thus we write

$$\sum_{0\le i,j\le n} \binom{i}{j} F_{i+j} = \sum_{0\le i,j\le 4k} \binom{i}{j} F_{i+j} = \frac{1}{\alpha-\beta} \sum_{0\le i,j\le 4k} \binom{i}{j} (\alpha^{i+j} - \beta^{i+j})$$
$$= \frac{1}{\alpha-\beta} \left[\sum_{0\le i,j\le 4k} \binom{i}{j} \alpha^{i+j} - \sum_{0\le i,j\le 4k} \binom{i}{j} \beta^{i+j} \right],$$

which, by Lemma 1, equals

$$\frac{1}{\alpha-\beta} \left[\frac{\left(\alpha+\alpha^2\right)^{4k+1} - 1}{\alpha+\alpha^2-1} - \frac{\left(\beta+\beta^2\right)^{4k+1} - 1}{\beta+\beta^2-1} \right],$$

which, by $\alpha^2 = \alpha+1,\ \alpha^2+\alpha-1 = 2\alpha$, $\beta^2 = \beta+1$ and $\beta^2+\beta-1 = 2\beta$, equals

$$\frac{1}{\alpha-\beta} \left[\frac{\alpha^{12k+3}-1}{2\alpha} - \frac{\beta^{12k+3}-1}{2\beta} \right]$$
$$= -\frac{1}{2(\alpha-\beta)} (-\alpha^{12k+2} + \beta^{12k+2} + \alpha - \beta)$$
$$= \frac{1}{2} \left[\frac{\alpha^{12k+2} - \beta^{12k+2}}{\alpha-\beta} - 1 \right] = \frac{1}{2}(F_{12k+2} - 1),$$

which, by Lemma 3, with the case $m = 6k$ *and* $n = 6k+2$, gives us

$$\sum_{0\le i,j\le 4k} \binom{i}{j} F_{i+j} = \frac{1}{2} F_{6k} L_{6k+2}$$

or, for $n = 4k$,

$$\sum_{0\le i,j\le n} \binom{i}{j} F_{i+j} = \frac{1}{2} F_{\frac{3n}{2}} L_{\frac{3n+4}{2}},$$

as claimed.

Now we prove the third identity. Similarly we write

$$\sum_{0\le i,j\le n} \binom{i}{j} F_{(4t+2)i+j} = \frac{1}{\alpha-\beta} \sum_{0\le i,j\le n} \binom{i}{j} \left(\alpha^{(4t+2)i+j} - \beta^{(4t+2)i+j} \right)$$
$$= \frac{1}{\alpha-\beta} \left[\frac{\left(\alpha^{4t+2}+\alpha^{4t+3}\right)^{n+1} - 1}{\alpha^{4t+2}+\alpha^{4t+3}-1} - \frac{\left(\beta^{4t+2}+\beta^{4t+3}\right)^{n+1} - 1}{\beta^{4t+2}+\beta^{4t+3}-1} \right]$$

which, since $\alpha^2 = \alpha+1$ and $\beta^2 = \beta+1$, equals

$$\frac{1}{\alpha-\beta} \left[\frac{\alpha^{(4t+4)(n+1)}-1}{\alpha^{4t+4}-1} - \frac{\beta^{(4t+4)(n+1)}-1}{\beta^{4t+4}-1} \right]$$

$$= \frac{1}{\alpha-\beta}$$
$$\times \frac{\beta^{4tn+4t+4n+4} - \alpha^{4tn+4t+4n+4} + \alpha^{4tn+4n} - \beta^{4tn+4n} + \alpha^{4t+4} - \beta^{4t+4}}{2-(\alpha^{4t+4}+\beta^{4t+4})}$$
$$= \frac{1}{2-L_{4t+4}}(-F_{4tn+4t+4n+4} + F_{4tn+4n} + F_{4t+4})$$
$$= \frac{1}{L_{4t+4}-2}(F_{4tn+4t+4n+4} - F_{4tn+4n} - F_{4t+4})$$
$$= \frac{1}{L_{4t+4}-2}\left(F_{4(t+1)(n+1)} - F_{4n(t+1)} - F_{4(t+1)}\right),$$

which, by Lemma 2, equals

$$\frac{5F_{2(t+1)(n+1)}F_{2n(t+1)}F_{2(t+1)}}{L_{4t+4}-2},$$

which, by Lemma 4, gives us the claim as

$$\sum_{0\le i,j\le n} \binom{i}{j} F_{2(2t+1)i+j} = \frac{F_{2n(t+1)}F_{2(n+1)(t+1)}}{F_{2(t+1)}}.$$

For the others, we only give some hints. The proofs of (2), (6) and (7) follow from Lemmas 1,2 and 4. The proofs of (4) and (5) follow from Lemmas 1 and 3. □

3. Alternating analogues of Binomial-Double-Sums

In this section, we present various alternating binomial double sums including the Fibonacci numbers. Now we continue to give some auxiliary Fibonacci-Lucas identities. As a showcase, we only prove Lemma 7. The others could be easily and similarly proven.

Lemma 5. *For odd integer m and even integer n,*

$$F_{2(m+n)} - F_{2m} + F_{2n} = 5F_mF_nF_{m+n}.$$

Lemma 6. *For even integers m and n,*

$$F_{2(m+n)} + F_{2m} - F_{2n} = F_mL_nL_{m+n},$$

Lemma 7. *For integers m and n,*

$$F_{2(m+n)} + F_{2m} + F_{2n} = \begin{cases} 5F_mF_nF_{m+n} & \text{if } m \text{ and } n \text{ are odd,} \\ L_mL_nF_{m+n} & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

Proof. We only give the proof for the case m and n are odd. Consider the RHS of the claim by the Binet formula and $\alpha\beta = -1$ for odd integers m and n,

$$5F_mF_nF_{m+n}$$

$$= \frac{5(\alpha^m - \beta^m)(\alpha^n - \beta^n)(\alpha^{m+n} - \beta^{m+n})}{(\alpha-\beta)^3}$$
$$= \frac{(\alpha^{m+n} + \beta^{m+n} - (\alpha\beta)^m(\beta^{n-m} + \alpha^{n-m}))(\alpha^{m+n} - \beta^{m+n})}{\alpha-\beta}$$
$$= \frac{\alpha^{2m+2n} - \beta^{2m+2n} - (-1)^m(\alpha^{2n} - \beta^{2n} + (\alpha\beta)^n(\beta^{-m}\alpha^m - \alpha^{-m}\beta^m))}{\alpha-\beta}$$
$$= \frac{\alpha^{2m+2n} - \beta^{2m+2n} - (-1)^m(\alpha^{2n} - \beta^{2n} - (-1)^n(\alpha^{2m} - \beta^{2m}))}{\alpha-\beta}$$
$$= \frac{\alpha^{2m+2n} - \beta^{2m+2n} + \alpha^{2n} - \beta^{2n} + \alpha^{2m} - \beta^{2m}}{\alpha-\beta}$$
$$= F_{2m+2n} + F_{2n} + F_{2m},$$

as claimed. □

Now we shall give our main result.

Theorem 2. *For all nonnegative integer n and any integer t,*

(1)

$$\sum_{0\le i,j\le n} \binom{i}{j}(-1)^i F_{i+j} = \frac{(-1)^n}{2}\begin{cases} F_{\frac{3n}{2}} L_{\frac{3n+2}{2}} & \text{if } n \equiv 0 (\text{mod } 4), \\ F_{\frac{3n-1}{2}} L_{\frac{3(n+1)}{2}} & \text{if } n \equiv 1 (\text{mod } 4), \\ L_{\frac{3n}{2}} F_{\frac{3n+2}{2}} & \text{if } n \equiv 2 (\text{mod } 4), \\ L_{\frac{3n-1}{2}} F_{\frac{3(n+1)}{2}} & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

(2)

$$\sum_{0\le i,j\le n} \binom{i}{j}(-1)^i F_{4ti+j} = \frac{(-1)^n F_{(2t+1)n} F_{(2t+1)(n+1)}}{F_{2t+1}}.$$

(3)

$$\sum_{0\le i,j\le n} \binom{i}{j}(-1)^i F_{(4t+2)i+j}$$
$$= \frac{(-1)^n}{L_{2(t+1)}}\begin{cases} L_{2(n+1)(t+1)} F_{2n(t+1)} & \text{if } n \text{ is even}, \\ F_{2(n+1)(t+1)} L_{2n(t+1)} & \text{if } n \text{ is odd}. \end{cases}$$

(4)

$$\sum_{0\le i,j\le n} \binom{i}{j}(-1)^i F_j = (-1)^n F_n F_{n+1}.$$

(5)

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^i F_{2i-j}=\frac{(-1)^n}{2}\begin{cases} F_{\frac{3n}{2}}L_{\frac{3n+2}{2}} & \text{if } n\equiv 0(\text{mod } 4),\\ F_{\frac{3n-1}{2}}L_{\frac{3(n+1)}{2}} & \text{if } n\equiv 1(\text{mod } 4),\\ L_{\frac{3n}{2}}F_{\frac{3n+2}{2}} & \text{if } n\equiv 2(\text{mod } 4),\\ L_{\frac{3n-1}{2}}F_{\frac{3(n+1)}{2}} & \text{if } n\equiv 3(\text{mod } 4).\end{cases}$$

(6)

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^i F_{(4t+1)i-j}=\frac{(-1)^n F_{(2t+1)n}F_{(2t+1)(n+1)}}{F_{2t+1}}.$$

(7)

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^i F_{(4t+3)i-j}$$
$$=\frac{(-1)^n}{L_{2(t+1)}}\begin{cases} L_{2(n+1)(t+1)}F_{2n(t+1)} & \text{if } n \text{ is even},\\ F_{2(n+1)(t+1)}L_{2n(t+1)} & \text{if } n \text{ is odd}.\end{cases}$$

Proof. We only prove the second identity. Consider

$$\sum_{0\le i,j\le n}\binom{i}{j}(-1)^i F_{4ti+j}=\frac{1}{\alpha-\beta}\sum_{0\le i,j\le n}\binom{i}{j}(-1)^i(\alpha^{4ti+j}-\beta^{4ti+j})$$
$$=\frac{1}{\alpha-\beta}\left[\sum_{0\le i,j\le n}\binom{i}{j}(-1)^i\alpha^{4ti+j}-\sum_{0\le i,j\le n}\binom{i}{j}(-1)^i\beta^{4ti+j}\right],$$

which, by Eq. (2.1), equals

$$\frac{1}{\alpha-\beta}\left[\frac{(-1)^n\left(\alpha^{4t}+\alpha^{4t+1}\right)^{n+1}+1}{\alpha^{4t}+\alpha^{4t+1}+1}-\frac{(-1)^n\left(\beta^{4t}+\beta^{4t+1}\right)^{n+1}+1}{\beta^{4t}+\beta^{4t+1}+1}\right],$$

which, by $\alpha^2=\alpha+1$ and $\beta^2=\beta+1$, equals

$$\frac{(-1)^n}{\alpha-\beta}\left[\frac{\alpha^{(4t+2)(n+1)}+(-1)^n}{\alpha^{4t+2}+1}-\frac{\beta^{(4t+2)(n+1)}+(-1)^n}{\beta^{4t+2}+1}\right]$$
$$=\frac{(-1)^n}{\alpha-\beta}\left[\frac{\alpha^{4tn+4t+2n+2}+(-1)^n}{\alpha^{4t+2}+1}-\frac{\beta^{4tn+4t+2n+2}+(-1)^n}{\beta^{4t+2}+1}\right]$$
$$=\frac{(-1)^n}{\left(\alpha^{4t+2}+\beta^{4t+2}+2\right)(\alpha-\beta)}$$
$$\times\left(\alpha^{4tn+2n}-\beta^{4tn+2n}+(-1)^n\beta^{4t+2}\right.$$
$$\left.-(-1)^n\alpha^{4t+2}+\alpha^{4tn+4t+2n+2}-\beta^{4tn+4t+2n+2}\right),$$

which, by the Binet formulæ of $\{F_n, L_n\}$, equals

$$\frac{(-1)^n}{L_{4t+2}+2}(F_{4tn+4t+2n+2} - (-1)^n F_{4t+2} + F_{4tn+2n}),$$

which, by Lemmas 7,5 and 4, equals

$$\frac{5(-1)^n}{L_{4t+2}+2} F_{2tn+2t+n+1} F_{2t+1} F_{2tn+n} = \frac{(-1)^n}{F_{2t+1}} F_{(2t+1)n} F_{(2t+1)(n+1)},$$

as claimed.

The others could be similarly proven. We only give hints for them. The proofs of (1) and (5) follow from Eq. (2.1) and Lemma 3. The proofs of (3) and (7) follow from Eq. (2.1), and, Lemmas 4,7 and 6. The proof of (4) follows from Eq. (2.1), and, Lemmas 7 and 5. The proof of (6) follows from Eq. (2.1), and, Lemmas 4,7 and 5. □

Now, by using Eq. (2.2) and Lemma 3 , we give our other result without proof.

Theorem 3. *For nonnegative integer n,*

(1)

$$\sum_{0\le i,j\le n} \binom{i}{j} (-1)^j F_j = -\begin{cases} F_{\frac{n}{2}} L_{\frac{n+4}{2}} & \text{if } n \equiv 0 (\text{mod } 4),\\ F_{\frac{n+3}{2}} L_{\frac{n+1}{2}} & \text{if } n \equiv 1 (\text{mod } 4),\\ L_{\frac{n}{2}} F_{\frac{n+4}{2}} & \text{if } n \equiv 2 (\text{mod } 4),\\ L_{\frac{n+3}{2}} F_{\frac{n+1}{2}} & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

(2) *i)*

$$\sum_{0\le i,j\le n} \binom{i}{j} (-1)^j F_{i+j} = 0.$$

ii)

$$\sum_{0\le i,j\le n} \binom{i}{j} (-1)^j F_{2i-j} = 0.$$

By using Eq. (2.3) and Lemma 3, we give our last result without proof.

Theorem 4. *For nonnegative integer n,*

(1)

$$\sum_{0\le i,j\le n} \binom{i}{j} (-1)^{i+j} F_j = (-1)^{n+1} \begin{cases} F_{\frac{n}{2}} L_{\frac{n-2}{2}} & \text{if } n \equiv 0 (\text{mod } 4),\\ F_{\frac{n-3}{2}} L_{\frac{n+1}{2}} & \text{if } n \equiv 1 (\text{mod } 4),\\ L_{\frac{n}{2}} F_{\frac{n-2}{2}} & \text{if } n \equiv 2 (\text{mod } 4),\\ L_{\frac{n-3}{2}} F_{\frac{n+1}{2}} & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

(2) *i)*

$$\sum_{0\leq i,j\leq n}\binom{i}{j}(-1)^{i+j}F_{i+j}=0.$$

ii)

$$\sum_{0\leq i,j\leq n}\binom{i}{j}(-1)^{i+j}F_{2i-j}=0.$$

As a final note, we would like to mention that we frequently compute the sums including Fibonacci numbers rather than Lucas numbers. We leave to compute sums including the Lucas numbers. We hope that such sums have nice multiplication forms that could be found and added to the literature.

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