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A new type of Sylvester–Kac matrix and its spectrum

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ABSTRACT

The Sylvester–Kac matrix, sometimes known as Clement matrix, has many extensions and applications throughout more than a century of its existence. The computation of the eigenvalues or even the determinant have always been challenging problems. In this paper, we aim the introduction of a new family of a Sylvester–Kac type matrix and evaluate the corresponding spectrum. As a consequence, we establish a formula for the determinant.

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1. Introduction

The Sylvester–Kac matrix, also known as Clement matrix, is the $(n+1)\times(n+1)$ tridiagonal matrix with zero main diagonal, one subdiagonal $(1, 2, \dots, n)$, while the other one stands in the reversed order, i.e.

$$A_n = \begin{pmatrix} 0 & 1 & & & & \\ n & 0 & 2 & & & \\ & n-1 & \ddots & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & 0 & n \\ & & & & 1 & 0 \end{pmatrix}.$$

The British mathematician James Joseph Sylvester was the first to consider this matrix in 1854 in his short communication [1], conjecturing that the determinant of its characteristic

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matrix

$$A_n(x) = \begin{pmatrix} x & 1 & & & & \\ n & x & 2 & & & \\ & n-1 & \ddots & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & x & n \\ & & & & 1 & x \end{pmatrix}$$

was

$$\det A_n(x) = \prod_{k=0}^{n} (x + n - 2k). \tag{1}$$

The first proof of Sylvester's determinantal formula is attributed by Muir to Francesco Mazza in 1866 [2, pp. 442], with a small typographical error as noticed in [3]. Nowadays it is consensual that Mark Kac, in 1947, with his Chauvenet prize-winning paper [4], was, in fact, the first to fully prove the formula, using the method of generating functions, and to provide a polynomial characterization of the eigenvectors. For some early history of this, the reader is referred to [5]. Results on the spectrum were scrutinized, independently rediscovered, and extended by many authors based on different approaches [5–16].

Recently, a new interest emerged in the literature about the Sylvester–Kac matrix, with many new extensions and major results. Perhaps the most relevant can be found in [3, 17–20].

In [18], E. Kılıç and T. Arıkan proposed an extension of $A_n(x)$, namely

$$A_n(x,y) = \begin{pmatrix} x & 1 & & & & \\ n & y & 2 & & & \\ & n-1 & x & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & y & n \\ & & & & 1 & x \end{pmatrix}$$

if n is even, and

$$A_n(x,y) = \begin{pmatrix} x & 1 & & & \\ n & y & 2 & & \\ & n-1 & x & \ddots & \\ & & \ddots & \ddots & n \\ & & & 1 & y \end{pmatrix},$$

otherwise, and explicitly evaluate its spectrum, say $\lambda(A_n(x,y))$, using some similarity techniques:

$$\lambda(A_n(x,y)) = \left\{ \frac{1}{2}(x+y) \mp \frac{1}{2}\sqrt{(x-y)^2 + (4k)^2} \right\}_{k=1}^{n/2} \cup \{x\}, \quad \text{for } n \text{ even},$$

and

$$\lambda(A_n(x,y)) = \left\{\frac{1}{2}(x+y) \mp \frac{1}{2}\sqrt{(x-y)^2 + (4k+2)^2}\right\}_{k=0}^{(n-1)/2}, \quad \text{for } n \text{ odd.}$$

The determinant now follows.

Theorem 1.1 ([18]): *The determinant of $A_n(x,y)$ is*

$$\det A_n(x,y) = \begin{cases} x\prod_{t=1}^{n/2}(xy-(2t)^2), & \text{if } n \text{ is even,} \\ \prod_{t=0}^{(n-1)/2}(xy-(2t+1)^2), & \text{if } n \text{ is odd.} \end{cases}$$

This extension is in the spirit of the original claim proposed Sylvester since he explicitly conjectured the determinant of $A_n(x)$. The matrix $A_n(x,y)$ is also an extension of a previous work by E. Kılıç [20], where $y = -x$.

In this paper, we aim the introduction of a new type of Sylvester–Kac matrix, denoted by $G_n(x)$ or, briefly, G_n,

$$G_n(x) = \begin{pmatrix} x & n+3 & & & & \\ n & x & n+4 & & & \\ & n-1 & x & \ddots & & \\ & & \ddots & \ddots & 2n+1 & \\ & & & 2 & x & 2n+2 \\ & & & & 1 & x \end{pmatrix}_{(n+1)\times(n+1)}, \quad (2)$$

and determine its spectrum, which we will denote by $\lambda(G_n)$. Then we formulate its determinant. Some consequences will be presented as well. In the end, we establish a generalization of the matrix $G_n(x)$, which we will denote by $G_n(x,y)$. Here the main diagonal entries will be in a 2-periodic form, oscillating between x to y. Setting $x = y$, we will recover the matrix $G_n(x)$. Notice that all these matrices are of order $n+1$.

2. The spectrum of $G_n(x)$

In this section we first find the spectrum of $G_n(x)$, denoted by $\lambda(G_n(x))$ and, later on, derive its determinant.

Theorem 2.1: *The eigenvalues of $G_n(x)$ are given by*

$$\begin{aligned}\lambda(G_{2n-1}) &= \{x \pm 2, x \pm 6, x \pm 10, \ldots, x \pm 2\,(2n-1)\} \\ &= \{x \pm 2\,(2k-1)\}_{k=1}^{n}\end{aligned}$$

and

$$\begin{aligned}\lambda(G_{2n}) &= \{x, x \pm 4, x \pm 8, x \pm 12, \ldots, x \pm 4n\} \\ &= \{x \pm 2\,(2k)\}_{k=0}^{n}.\end{aligned}$$

We start finding two eigenvalues of G_n and then two corresponding left eigenvectors associated them.

Let us define the two $2n+1$-vectors,

$$u_1 = (1, 2, 3, \ldots, 2n+1) \quad \text{and} \quad u_2 = (1, -2, 3, \ldots, -2n, 2n+1).$$

The next lemma says that u_1 and u_2 are eigenvectors of G_{2n}.

Lemma 2.2: *The matrix G_{2n} has the eigenvalues $\lambda^+ = x + 4n$ and $\lambda^- = x - 4n$ with left eigenvectors u_1 and u_2, respectively.*

Proof: To prove our claim, it is sufficient to show that

$$u_1 G_{2n} = \lambda^+ u_1 \quad \text{and} \quad u_2 G_{2n} = \lambda^- u_2 \,.$$

Notice the kth component of u_1 by is precisely k. From the definitions of G_{2n} and u_1, we should show that

$$\begin{aligned} x + (2n)2 &= \lambda^+, \\ (4n+2)\,2n + x(2n+1) &= \lambda^+(2n+1), \\ (k-1)(2n+1+k) + kx + (k+1)(2n+1-k) &= \lambda^+ k, \quad \text{for } 2 \leqslant k \leqslant 2n-1. \end{aligned} \tag{3}$$

The only equalities requiring some algebra are those defined in (3). Our first claim follows then.

The other case, i.e. $u_2 G_{2n} = \lambda^- u_2$, can be handled in a similar way. ■

Similarly to the previous case, we define two $2n$-vectors:

$$v_1 = (1, 2, 3, \ldots, 2n) \quad \text{and} \quad v_2 = (1, -2, 3, \ldots, -2n).$$

The next lemma can be proved analogously to the previous result.

Lemma 2.3: *The matrix G_{2n-1} has the eigenvalues $\mu^+ = x + 2(2n-1)$ and $\mu^- = x - 2(2n-1)$ with left eigenvectors v_1 and v_2, respectively.*

Now our purpose is to find similar matrices to G_{2n} and G_{2n-1}, respectively. We start with the matrix G_{2n}.

Define a matrix T of order $2n+1$ as shown

$$T = \left(\begin{array}{cc|ccccc} 1 & 2 & 3 & \cdots & 2n & 2n+1 \\ 1 & -2 & 3 & \cdots & -2n & 2n+1 \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{(2n-1)\times 2}} & \multicolumn{4}{c}{I_{2n-1}} \end{array} \right),$$

where $\mathbf{0}_{m\times n}$ is the $m \times n$ zero matrix and I_k is the identity matrix of order k. Its inverse is

$$T^{-1} = \left(\begin{array}{cc|cccccccc} \frac{1}{2} & \frac{1}{2} & -3 & 0 & -5 & 0 & \cdots & 0 & -(2n+1) \\ \frac{1}{4} & -\frac{1}{4} & 0 & -2 & 0 & -3 & \cdots & -n & 0 \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{(2n-1)\times 2}} & \multicolumn{7}{c}{I_{2n-1}} \end{array} \right).$$

We can easily check that G_{2n} is similar to the matrix

$$E = \left(\begin{array}{cc|c} \lambda^+ & 0 & \mathbf{0}_{2\times(2n-1)} \\ 0 & \lambda^- & \\ \hline \dfrac{2n-1}{4} & -\dfrac{2n-1}{4} & \\ \mathbf{0}_{(2n-2)\times 2} & & W \end{array}\right),$$

where W is the matrix of order $2n-1$ is given by

$$W = \begin{pmatrix} x & 7-2n & 0 & -3(2n-1) & \cdots & 0 & -n(2n-1) & 0 \\ 2n-2 & x & 2n+6 & 0 & & & & \\ & 2n-3 & x & 2n+7 & \ddots & & & \\ & & 2n-4 & \ddots & \ddots & 0 & & \\ & & & \ddots & \ddots & 4n & 0 & \\ & & & & 3 & x & 4n+1 & 0 \\ & & & & & 2 & x & 4n+2 \\ & & & & & & 1 & x \end{pmatrix},$$

since $E = TG_{2n}T^{-1}$. Consequently, $\lambda^{\pm}$ are eigenvalues of both E and G_{2n}.

We will focus now on the matrix G_{2n-1}. Define the matrix Y of order $2n$ as

$$Y = \left(\begin{array}{cc|cccc} 1 & 2 & 3 & \cdots & 2n-1 & 2n \\ 1 & -2 & 3 & \cdots & 2n-1 & -2n \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{(2n-2)\times 2}} & \multicolumn{4}{c}{I_{2n-2}} \end{array}\right).$$

Similarly to the previous case, we obtain have

$$Y^{-1} = \left(\begin{array}{cc|cccccccc} \frac{1}{2} & \frac{1}{2} & -3 & 0 & -5 & 0 & \cdots & 0 & -(2n-1) & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & -2 & 0 & -3 & \cdots & -(n-1) & 0 & -n \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{(2n-2)\times 2}} & \multicolumn{8}{c}{I_{2n-2}} \end{array}\right).$$

Therefore, G_{2n-1} is similar, via Y , to the matrix $D = YG_{2n-1}Y^{-1}$ of the form

$$D = \left(\begin{array}{cc|c} \mu^+ & 0 & \mathbf{0}_{2\times(2n-2)} \\ 0 & \mu^- & \\ \hline \dfrac{n-1}{2} & -\dfrac{n-1}{2} & \\ \mathbf{0}_{(2n-3)\times 2} & & Q \end{array}\right),$$

where Q is the matrix, of order $2n-2$,

$$Q = \begin{pmatrix} x & -2(n-4) & 0 & -6(n-1) & \cdots & 0 & -2n(n-1) \\ 2n-3 & x & 2n+5 & 0 & & & \\ & 2n-4 & x & 2n+6 & \ddots & & \\ & & 2n-5 & \ddots & \ddots & 0 & \\ & & & \ddots & \ddots & 4n-1 & 0 \\ & & & & 2 & x & 4n \\ & & & & & 1 & x \end{pmatrix}.$$

Thus μ^+ and μ^- are eigenvalues of the matrix G_{2n-1}.

To compute the remaining eigenvalues of G_{2n-1} and G_{2n}, we proceed providing some auxiliary results.

Define an upper triangle matrix U_n as follows

$$U_{2\ell-1} = \begin{pmatrix} 1 & 0 & 2 & 0 & 3 & \cdots & 0 & \ell \\ & 1 & 0 & 2 & 0 & 3 & \ddots & 0 \\ & & 1 & 0 & 2 & 0 & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & 3 \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & \ddots & \ddots & 2 \\ & & & & & & \ddots & 0 \\ & & & & & & & 1 \end{pmatrix}_{(2\ell-1)\times(2\ell-1)}$$

and

$$U_{2\ell} = \begin{pmatrix} 1 & 0 & 2 & 0 & 3 & 0 & \cdots & \ell & 0 \\ & 1 & 0 & 2 & 0 & 3 & 0 & \ddots & \ell \\ & & 1 & 0 & 2 & 0 & 3 & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & & \ddots & \ddots & \ddots & \ddots & 3 \\ & & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & & \ddots & \ddots & 2 \\ & & & & & & & \ddots & 0 \\ & & & & & & & & 1 \end{pmatrix}_{2\ell\times 2\ell}.$$

Therefore, for any parity of n, the inverse matrix U_n^{-1} is

$$U_n^{-1} = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 & & & \\ & 1 & 0 & -2 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & -2 & 0 & 1 \\ & & & & 1 & 0 & -2 & 0 \\ & & & & & 1 & 0 & -2 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix}$$

Taking into account the definition of U_n, we clearly have

$$G_{2n-2} = U_{2n-1} W U_{2n-1}^{-1} \quad \text{and} \quad G_{2n-1} = U_{2n} Q U_{2n}^{-1}.$$

Furthermore, let us define the following matrix of order n

$$M_n = \left(\begin{array}{c|c} I_2 & \mathbf{0}_{2\times(n-2)} \\ \hline \mathbf{0}_{(n-2)\times 2} & U_{n-2} \end{array}\right).$$

Hence we get

$$M_{2n+1}^{-1} E M_{2n+1} = \left(\begin{array}{cc|c} \lambda^+ & 0 & \mathbf{0}_{2\times(2n-1)} \\ 0 & \lambda^- & \\ \hline \frac{2n-1}{4} & -\frac{2n-1}{4} & \\ \mathbf{0}_{(2n-2)\times 2} & & U_{2n-1}^{-1} W U_{2n-1} \end{array}\right)$$

and

$$M_{2n}^{-1} D M_{2n} = \left(\begin{array}{cc|c} \mu^+ & 0 & \mathbf{0}_{2\times(2n-2)} \\ 0 & \mu^- & \\ \hline \frac{n-1}{2} & -\frac{n-1}{2} & \\ \mathbf{0}_{(2n-3)\times 2} & & U_{2n-2}^{-1} Q U_{2n-2} \end{array}\right).$$

Up to now, we derived the identities

$$\begin{aligned} E &= T\, G_{2n}\, T^{-1}, \\ D &= Y\, G_{2n-1}\, Y^{-1}, \\ G_{2n-2} &= U_{2n-1}\, W\, U_{2n-1}^{-1}, \\ G_{2n-1} &= U_{2n-2}\, Q\, U_{2n-2}^{-1}. \end{aligned}$$

From the definition of G_n given in (2), both $M_{2n+1}^{-1} E M_{2n+1}$ and $M_{2n}^{-1} D M_{2n}$ can be rewritten in the following lower-triangular block form

$$\left(\begin{array}{cc|c} \lambda^+ & 0 & \\ 0 & \lambda^- & 0 \\ \hline * & & G_{2n-1} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} \mu^+ & 0 & \\ 0 & \mu^- & 0 \\ \hline * & & G_{2n-2} \end{array}\right), \tag{4}$$

respectively.

From (4), we get the recurrences on $n > 0$,

$$\det G_{2n-1} = \mu^+\mu^- \det G_{2n-3} = (x^2 - 4(2n-1)^2) G_{2n-3}, \quad \text{with } \det G_{-1} = 1$$

and

$$\det G_{2n} = \lambda^+\lambda^- \det G_{2n-2} = (x^2 - 16n^2) \det G_{2n-2}, \quad \text{with } \det G_0 = x,$$

which means that

$$\det G_n = (x^2 - 4n^2) \det G_{n-2},$$

with the two initial conditions stated above. Finally, we obtain Theorem 2.1.

Now the determinant of G_n follows immediately.

Theorem 2.4: *The determinant of* $G_n(x)$ *is*

$$\det G_n(x) = \begin{cases} \displaystyle\prod_{t=1}^{(n+1)/2} (x^2 - (4t-2)^2), & \text{if } n \text{ is odd,} \\ \displaystyle x\prod_{t=0}^{n/2} (x^2 - (4t)^2), & \text{if } n \text{ is even.} \end{cases}$$

3. A generalization for $G_n(x)$

In the section, we will discuss a generalization of $G_n(x)$, where the main diagonal is biperiodic, as described in the introduction and studied in a related problem in [18].

Let us consider a new matrix $G_n(x, y)$ defined as

$$G_n(x,y) = \begin{pmatrix} x & n+3 & & & & & & \\ n & y & n+4 & & & & & \\ & n-1 & x & n+5 & & & & \\ & & n-2 & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \ddots & y & 2n+1 & \\ & & & & & 2 & x & 2n+2 \\ & & & & & & 1 & y \end{pmatrix}$$

if n is odd, and

$$G_n(x,y) = \begin{pmatrix} x & n+3 & & & & & & \\ n & y & n+4 & & & & & \\ & n-1 & x & n+5 & & & & \\ & & n-2 & y & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \ddots & x & 2n+1 & \\ & & & & & 2 & y & 2n+2 \\ & & & & & & 1 & x \end{pmatrix},$$

if n is even.

Now for later use, we shall note a fact. Consider

$$F_{n+1} := (\sqrt{z})^{n+1} \det \begin{pmatrix} \sqrt{z} & b_1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_n \\ & & c_{n-1} & \sqrt{z} \end{pmatrix} = \det \begin{pmatrix} z & \sqrt{z}b_1 & & \\ \sqrt{z}c_1 & \ddots & \ddots & \\ & \ddots & \ddots & \sqrt{z}b_n \\ & & \sqrt{z}c_{n-1} & z \end{pmatrix},$$

which, by expanding with the Laplace expansion according to the last row or column, gives us

$$F_{n+1} = zF_n - zb_nc_nF_{n-1} \quad \text{with initials } F_0 = 1 \quad \text{and} \quad F_1 = z.$$

Meanwhile, now consider

$$P_{n+1} := \det \begin{pmatrix} z & b_1 & & \\ zc_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_n \\ & & zc_n & z \end{pmatrix}$$

and if we expand it according to the last row or column, we obtain

$$P_{n+1} = zP_n - zb_nc_nP_{n-1} \quad \text{with } P_0 = 1 \quad \text{and } P_1 = z.$$

Thus we deduce the fact that since the sequences $\{F_n\}$ and $\{P_n\}$ have the same recursions and the same initials, these are the same. Clearly, we have

$$\det \begin{pmatrix} z & b_1 & & \\ zc_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_n \\ & & zc_n & z \end{pmatrix} = (\sqrt{z})^{n+1} \det \begin{pmatrix} \sqrt{z} & b_1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_n \\ & & c_{n-1} & \sqrt{z} \end{pmatrix}.$$

On the other hand, we also obtain similar determinantal identity as shown

$$\begin{aligned} &(xy)^{\lfloor n+1/2 \rfloor} y^{n+1 \bmod 2} \det G_n(x,y) \\ &= \det \begin{pmatrix} xy & n+3 & & & & \\ xyn & xy & n+4 & & & \\ & xy(n-1) & \ddots & \ddots & & \\ & & \ddots & \ddots & 2n+1 & \\ & & & xy \cdot 2 & xy & 2n+2 \\ & & & & xy \cdot 1 & xy \end{pmatrix}. \end{aligned} \tag{1}$$

We may prove this identity in a similar way to the previous one. In fact, again using a similar approach as for the previous equality according to the parity of n, the proof could be easily obtained. Combining the two previous equalities and setting $z = \sqrt{xy}$, we get

$$\det G_n(x,y) = \begin{cases} \det G_n(\sqrt{xy}), & \text{if } n \text{ is odd,} \\ \sqrt{\frac{x}{y}} \det G_n(\sqrt{xy}), & \text{if } n \text{ is even.} \end{cases}$$

This means, from Theorem 2.4,

$$\det G_n(x,y) = \begin{cases} \displaystyle\prod_{t=1}^{(n+1)/2} (xy - (4t-2)^2), & \text{if } n \text{ is odd,} \\ \displaystyle x \prod_{t=0}^{n/2} (xy - (4t)^2), & \text{if } n \text{ is even.} \end{cases}$$

As a conclusion, we can set the eigenvalues for $G_n(x,y)$.

Theorem 3.1: *The eigenvalues of $G_n(x,y)$ are*:

$$\lambda(G_{2n-1}(x,y)) = \left\{\frac{x+y}{2} \pm \frac{1}{2}\sqrt{(x-y)^2+16(2t-1)^2}\right\}_{t=1}^{n}$$

and

$$\lambda(G_{2n}(x,y)) = \{x\} \cup \left\{\frac{x+y}{2} \pm \frac{1}{2}\sqrt{(x-y)^2+16(2t)^2}\right\}_{t=1}^{n}.$$

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