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THE INTERESTING SPECTRAL INTERLACING PROPERTY FOR A CERTAIN TRIDIAGONAL MATRIX*

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Abstract. In this paper, a new tridiagonal matrix, whose eigenvalues are the same as the Sylvester-Kac matrix of the same order, is provided. The interest of this matrix relies also in that the spectrum of a principal submatrix is also of a Sylvester-Kac matrix given rise to an interesting spectral interlacing property. It is proved alternatively that the initial matrix is similar to the Sylvester-Kac matrix.

Key words. Sylvester-Kac matrix, Tridiagonal matrices, Determinant, Eigenvalues.

AMS subject classifications. 15A15, 15A18, 65F15.

1. Introduction. For any positive integer n, the $n+1$ numbers

$$-n,\, -n+2,\, -n+4,\, \ldots,\, n-4,\, n-2,\, n \tag{1.1}$$

are the eigenvalues of the so-called Sylvester-Kac matrix

$$A_n = \begin{pmatrix} 0 & 1 & & & & \\ n & 0 & 2 & & & \\ & n-1 & \ddots & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & 0 & n \\ & & & & 1 & 0 \end{pmatrix}.$$

We will call the sequence (1.1) the *n-Sylvester spectrum*. In the matrices throughout the text, all non-mentioned entries should be read as zero.

The tridiagonal matrix A_n was first considered by J.J. Sylvester in 1854 in a succinct note [25] where its characteristic polynomial was conjectured. As many problems in mathematics, this was a simple problem to state but hard to prove. A definite proof to Sylvester's claim is commonly attributed to M. Kac (for both eigenvalues and eigenvectors) in his celebrated work [17], almost a century after the original statement. Notwithstanding, the Sylvester-Kac matrix has a rich history, with many proofs, in different areas, extensions, reinventions, and applications. Perhaps the most significant literature is [1, 3, 6, 7, 9–16, 18–24, 26, 27]. The Sylvester-Kac matrix, is also known as Clement matrix due to the independent study of P.A. Clement in [9].

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As mentioned in [26], there are many generalizations of Sylvester's claim. Some have been established by Askey and Wilson [2] and remain largely open. Interestingly and somehow surprisingly, there is a close connection with Krawtchouk polynomials, which are polynomials orthogonal with respect to a binomial distribution. On the other hand, there is also an intimate relation with graph theory, namely to problems about distance regular graphs [4, p. 246].

Let us consider the tridiagonal matrix

$$\tilde{H}_n = \begin{pmatrix} 0 & 1/2 & & & & \\ \sigma_{n,n} & 0 & 1/2 & & & \\ & \sigma_{n-1,n} & \ddots & \ddots & & \\ & & \ddots & \ddots & 1/2 & \\ & & & \sigma_{2,n} & 0 & 1 \\ & & & & \sigma_{1,n} & 0 \\ & & & & & \end{pmatrix},$$

where, for any $k = 0, 1, \ldots, n-1$, we define

$$\sigma_{k,n} = \frac{(n-k+1)(n+k)}{2}.$$

That is, $\sigma_{k,n}$ is the sum of all nonnegative integers from k to n. Clearly, the spectrum of $\tilde{H}_n$ is the same as the matrix

$$H_n = \begin{pmatrix} 0 & \frac{1}{2} & & & & \\ \frac{2n}{2} & 0 & \frac{2}{2} & & & \\ & \frac{2n-1}{2} & \ddots & \ddots & & \\ & & \ddots & \ddots & \frac{n-1}{2} & \\ & & & \frac{n+2}{2} & 0 & n \\ & & & & \frac{n+1}{2} & 0 \\ & & & & & \end{pmatrix}.$$

In this paper, we show by two distinct ways that H_n shares the same spectrum as the Sylvester-Kac matrix A_n, i.e., the n-Sylvester spectrum. What is particularly interesting in this matrix is that when we delete the last row and column of H_n, we get a principal submatrix whose eigenvalues form the $(n-1)$-Sylvester spectrum. This means that Cauchy's interlacing theorem satisfies

$$-n < 1-n < 2-n < \cdots < n-2 < n-1 < n.$$

Since we know all the spectral properties of the Sylvester-Kac matrix, this matrix is very useful as what is known as *test matrix*. In general, test matrices are used to evaluate the accuracy of matrix inversion programs since the exact inverses are known (cf. e.g. [5, 21] and references therein). Recently, Coelho, Dimitrov, and Rakai in [8] suggested a method for a fast estimation of the largest eigenvalue of an asymmetric tridiagonal matrix. The proposed procedure was based on the power method and the computation of the square of the original matrix. Then they provided numerical results with simulations in `C/C++` implementation in order

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to demonstrate the effectiveness of the proposed method. They adopted the Sylvester-Kac test matrix [21] for comparing the power method and the proposed method performance. We also refer to [24] for further usage of test matrices. It is our purpose that the new matrix that we will present here and the corresponding explicit eigenvalues will make a significant contribution these type of special matrices.

2. The spectrum of H_n. In this section we prove our main result. We use basically the technique of the left eigenvectors of H_n and an inductive approach to reach our aim.

THEOREM 2.1. *The eigenvalues of H_n are (1.1), i.e.,*

$$\{-2\ell, -2\ell+2, \dots, -2, 0, 2, \dots, 2\ell-2, 2\ell\}$$

for $n = 2\ell$, and

$$\{-2\ell-1, -2\ell+1, -2\ell+3, \dots, -1, 1, \dots, 2\ell-3, 2\ell-1, 2\ell+1\}$$

for $n = 2\ell+1$.

We start by finding two eigenvalues of H_n and then the two corresponding left eigenvectors associated to each of them.

Let us define the two $(2n+1)$-vectors

$$u^+ = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

and

$$u^- = \begin{pmatrix} 1 & -1 & 1 & -1 & \cdots & -1 & 1 \end{pmatrix}.$$

The next lemma is crucial and it says that u^+ and u^- are both left eigenvectors of H_{2n}.

LEMMA 2.2. *The matrix H_{2n} has the eigenvalues $\lambda^+ = 2n$ and $\lambda^- = -2n$ with left eigenvectors u^+ and u^-, respectively.*

Proof. To prove our claim, it is sufficient to show that

$$u^+ H_{2n} = \lambda^+ u^+ \quad \text{and} \quad u^- H_{2n} = \lambda^- u^- .$$

From the definitions of H_{2n} and u^+, we should show that

$$\left(u^+ H_{2n}\right)_{1,1} = \left(\lambda^+ u^+\right)_{1,1},$$
$$\left(u^+ H_{2n}\right)_{1,2n+1} = \left(\lambda^+ u^+\right)_{1,2n+1}$$

and

$$\left(u^+ H_{2n}\right)_{1,m} = \left(\lambda^+ u^+\right)_{1,m}, \quad \text{for } 1 < m < 2n+1.$$

The first two claims are simple to check. For example, the first identity comes from

$$\left(u^+ H_{2n}\right)_{1,1} = 2n = \lambda^+ = \left(\lambda^+ u^+\right)_{1,1}.$$

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We now focus on the case $2 \le k \le 2n$. We consider

$$\left(u^{+}H_{2n}\right)_{1,k} = \frac{k}{2} + \frac{4n-k}{2} = 2n.$$

On the other hand, the definition of λ^{+} gives

$$\left(\lambda^{+}u^{+}\right)_{1,k} = 2n,$$

as claimed. The other case, i.e., $u^{-}H_{2n} = \lambda^{-}u^{-}$, can be handled in a similar fashion. □

Similarly to the previous case, we define two $(2n+2)$-vectors:

$$v^{+} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

and

$$v^{-} = \begin{pmatrix} 1 & -1 & 1 & -1 & \cdots & 1 & -1 \end{pmatrix}.$$

The next lemma can be proved analogously to the previous result.

Lemma 2.3. *The matrix H_{2n+1} has the eigenvalues $\mu^{+} = 2n+1$ and $\mu^{-} = -(2n+1)$ with left eigenvectors v^{+} and v^{-}, respectively.*

For later use, we define an upper triangle matrix U_n of order n with

$$U_{i,i} = \frac{(n - \lfloor i/2 \rfloor)(2n+1-2\lceil i/2 \rceil)}{\binom{n+1}{2}}, \quad \text{for } 1 \le i \le n$$

and

$$U_{i,i+2r} = \frac{(n-i)(2n+1)}{\binom{n+1}{2}}, \quad \text{for } 1 \le i \le n-2r \;\text{ and }\; 1 \le r \le \left\lfloor \frac{n-1}{2} \right\rfloor,$$

and 0, otherwise, where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ stand for the floor and ceiling functions, respectively.

For example, when $n = 10$, we have

$$U_{10} = \frac{21}{55}\begin{bmatrix}
\frac{190}{21} & 0 & 9 & 0 & 9 & 0 & 9 & 0 & 9 & 0 \\
0 & \frac{57}{7} & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8 \\
0 & 0 & \frac{51}{7} & 0 & 7 & 0 & 7 & 0 & 7 & 0 \\
0 & 0 & 0 & \frac{136}{21} & 0 & 6 & 0 & 6 & 0 & 6 \\
0 & 0 & 0 & 0 & \frac{40}{7} & 0 & 5 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 5 & 0 & 4 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{13}{3} & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{26}{7} & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{22}{7} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{55}{21}
\end{bmatrix}.$$

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For an odd case as $n=11$, we have

$$U_{11}=\frac{23}{66}\begin{pmatrix} \frac{231}{23} & 0 & 10 & 0 & 10 & 0 & 10 & 0 & 10 & 0 & 10\\ 0 & \frac{210}{23} & 0 & 9 & 0 & 9 & 0 & 9 & 0 & 9 & 0\\ 0 & 0 & \frac{190}{23} & 0 & 8 & 0 & 8 & 0 & 8 & 0 & 8\\ 0 & 0 & 0 & \frac{171}{23} & 0 & 7 & 0 & 7 & 0 & 7 & 0\\ 0 & 0 & 0 & 0 & \frac{153}{23} & 0 & 6 & 0 & 6 & 0 & 6\\ 0 & 0 & 0 & 0 & 0 & \frac{136}{23} & 0 & 5 & 0 & 5 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{120}{23} & 0 & 4 & 0 & 4\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{105}{23} & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{91}{23} & 0 & 2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{78}{23} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{66}{23} \end{pmatrix}.$$

A routine calculation lead us to the inverse matrix $U_n^{-1}=(C_{ij})$ with the recursions

$$\frac{C_{i,i}}{C_{i+1,i+1}}=\frac{2n-i-1}{2n-i+1}$$

for $1\le i\le n-1$, while, for $1\le r\le \left\lceil \frac{n-2}{2}\right\rceil$,

$$\frac{C_{i,i+2r}}{C_{i+1,i+2r+1}}=\frac{i+2}{i+2r}\times\frac{n-i}{n-i-1}\times\frac{2n-2\,(r+1)-i}{2n+1-i}$$

and 0, otherwise, where the initials $C_{11}=\frac{n+1}{4n-2}$, $C_{11}/C_{13}=(2n-3)\,/\,(2n+1)$ and

$$\frac{C_{1,2i+1}}{C_{1,2i+3}}=\frac{(n-i-1)\,(2n-2i-3)}{h_{i+1}},$$

for $1\le i\le \left\lfloor \frac{n-3}{2}\right\rfloor$, where h_n is the Hexagonal number defined by $h_n=n\,(2n-1)$.

Now our purpose is to find similar matrices to H_{2n} and H_{2n+1}, respectively. For this purpose, we shall give the following result.

Lemma 2.4. *The spectrum of matrix $H_n,\ \sigma(H_n)$, satisfy that*

$$\sigma(H_{2n})=\left\{\lambda^+,\lambda^-\right\}\cup\sigma(H_{2n-2})$$

and

$$\sigma(H_{2n+1})=\left\{\mu^+,\mu^-\right\}\cup\sigma(H_{2n-1}).$$

Proof. First, we consider the matrix H_{2n}. Define a matrix T of order $2n+1$ as shown

$$T=\left(\begin{array}{cc|ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1\\ 1 & -1 & 1 & -1 & 1 & -1 & \cdots & -1 & 1\\ \hline \multicolumn{2}{c|}{0_{(2n-1)\times 2}} & \multicolumn{7}{c}{I_{2n-1}} \end{array}\right),$$

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where $\mathbf{0}_{(2n-1)\times 2}$ is the $(2n-1)\times 2$ zero matrix and I_k is the identity matrix of order k. Its inverse is

$$T^{-1} = \left(\begin{array}{cc|cccccccc} \frac{1}{2} & \frac{1}{2} & -1 & 0 & -1 & 0 & -1 & \cdots & 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -1 & 0 & -1 & 0 & \cdots & -1 & 0 \\ \hline \multicolumn{2}{c|}{0_{(2n-1)\times 2}} & \multicolumn{8}{c}{I_{2n-1}} \end{array}\right).$$

We can easily check that H_{2n} is similar to the matrix

$$E = \left(\begin{array}{cc|c} \lambda^+ & 0 & \\ 0 & \lambda^- & \mathbf{0}_{2\times(2n-1)} \\ \hline \frac{2\lambda^+-1}{4} & \frac{2\lambda^-+1}{4} & \\ \multicolumn{2}{c|}{\mathbf{0}_{(2n-2)\times 2}} & W \end{array}\right),$$

where W is the block of order $2n-1$ defined by

$$W = \begin{pmatrix} 0 & \frac{4-4n}{2} & 0 & -\frac{4n-1}{2} & \cdots & 0 & -\frac{4n-1}{2} & 0 \\ \frac{4n-2}{2} & 0 & \frac{4}{2} & & & & & \\ & \frac{4n-3}{2} & 0 & \frac{5}{2} & & & & \\ & & \frac{4n-4}{2} & 0 & \ddots & & & \\ & & & \frac{4n-5}{2} & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \frac{2n-1}{2} & \\ & & & & & \frac{2n+2}{2} & 0 & 2n \\ & & & & & & \frac{2n+1}{2} & 0 \end{pmatrix},$$

since $E = TH_{2n}T^{-1}$. Consequently, $\lambda^{\pm}$ are eigenvalues of both E and H_{2n}.

We will study now the matrix H_{2n+1}. Define the matrix Y of order $2n+2$ as

$$Y = \left(\begin{array}{cc|cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & \cdots & 1 & -1 \\ \hline \multicolumn{2}{c|}{0_{2n\times 2}} & \multicolumn{7}{c}{I_{2n}} \end{array}\right).$$

Likewise to the previous case, we obtain

$$Y^{-1} = \left(\begin{array}{cc|cccccccc} \frac{1}{2} & \frac{1}{2} & -1 & 0 & -1 & 0 & -1 & \cdots & -1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -1 & 0 & -1 & 0 & \cdots & 0 & -1 \\ \hline \multicolumn{2}{c|}{0_{2n\times 2}} & \multicolumn{8}{c}{I_{2n}} \end{array}\right).$$

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Therefore, H_{2n+1} is similar to $D = YH_{2n+1}Y^{-1}$, where

$$D = \left(\begin{array}{cc|c} \mu^+ & 0 & \\ 0 & \mu^- & \mathbf{0}_{2\times 2n} \\ \hline \frac{2\mu^+-1}{4} & \frac{2\mu^-+1}{4} & \\ & \mathbf{0}_{(2n-1)\times 2} & Q \end{array}\right)$$

and Q is the matrix, of order $2n$,

$$Q = \begin{pmatrix} 0 & \frac{2-4n}{2} & 0 & -\frac{4n+1}{2} & 0 & -\frac{4n+1}{2} & \cdots & 0 & -\frac{4n+1}{2} \\ \frac{4n}{2} & 0 & \frac{4}{2} & & & & & & \\ & \frac{4n-1}{2} & 0 & \frac{5}{2} & & & & & \\ & & \frac{4n-2}{2} & 0 & \frac{6}{2} & & & & \\ & & & \frac{4n-3}{2} & 0 & \ddots & & & \\ & & & & \ddots & \ddots & \frac{2n-1}{2} & & \\ & & & & & \frac{2n+4}{2} & 0 & \frac{2n}{2} & \\ & & & & & & \frac{2n+3}{2} & 0 & 2n+1 \\ & & & & & & & \frac{2n+2}{2} & 0 \end{pmatrix}.$$

Thus, μ^+ and μ^- are eigenvalues of the matrix H_{2n+1}.

To compute the remaining eigenvalues of H_{2n+1} and H_{2n}, we proceed providing some auxiliary results.

Taking into account the definition of U_n, we clearly have

$$H_{2n-2} = U_{2n-1}W\,U_{2n-1}^{-1} \quad \text{and} \quad H_{2n-1} = U_{2n}\,Q\,U_{2n}^{-1}.$$

Furthermore, if we define the matrix of order n

$$M_n = \left(\begin{array}{c|c} I_2 & \mathbf{0}_{2\times(n-2)} \\ \hline \mathbf{0}_{(n-2)\times 2} & U_{n-2} \end{array}\right),$$

then we get

$$M_{2n+1}EM_{2n+1}^{-1} = \left(\begin{array}{cc|c} \lambda^+ & 0 & \mathbf{0}_{2\times(2n-1)} \\ 0 & \lambda^- & \\ \hline \frac{(4n-1)(4n-3)}{4n} & -\frac{(4n-1)(4n-3)}{4n} & \\ \mathbf{0}_{(2n-2)\times 2} & & U_{2n-1}^{-1}WU_{2n-1} \end{array}\right)$$

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and

$$M_{2n+2}DM_{2n+2}^{-1} = \left(\begin{array}{cc|c} \mu^+ & 0 & \mathbf{0}_{2\times(2n-2)} \\ 0 & \mu^- & \\ \hline \frac{(4n-1)(4n+1)}{2(2n+1)} & -\frac{(4n-1)(4n+1)}{2(2n+1)} & \\ \mathbf{0}_{(2n-1)\times 2} & & U_{2n}^{-1}QU_{2n} \end{array}\right).$$

So far, we derived the identities

$$\begin{aligned} E &= T\,H_{2n}\,T^{-1}\,, \\ D &= Y\,H_{2n+1}\,Y^{-1}\,, \\ H_{2n-2} &= U_{2n-1}W\,U_{2n-1}^{-1}\,, \\ H_{2n-1} &= U_{2n}\,Q\,U_{2n}^{-1}\,. \end{aligned}$$

From the definition of H_n, both $M_{2n+1}EM_{2n+1}^{-1}$ and $M_{2n+2}DM_{2n+2}^{-1}$ can be rewritten in the following lower-triangular block form

$$\left(\begin{array}{cc|c} \lambda^+ & 0 & \multirow{2}{*}{0} \\ 0 & \lambda^- & \\ \hline \multicolumn{2}{c|}{*} & H_{2n-2} \end{array}\right) \quad \text{and} \quad \left(\begin{array}{cc|c} \mu^+ & 0 & \multirow{2}{*}{0} \\ 0 & \mu^- & \\ \hline \multicolumn{2}{c|}{*} & H_{2n-1} \end{array}\right), \tag{2.2}$$

respectively, which give us the claimed results. ∎

From (2.2), we derive our main result, Theorem 2.1, on the spectra of the matrix H_n.

Also for the matrix $H_n\left(x\right)$ defined by

$$H_n\left(x\right) = \begin{pmatrix} x & \frac{1}{2} & & & & \\ \frac{2n}{2} & x & \frac{2}{2} & & & \\ & \frac{2n-1}{2} & \ddots & \ddots & & \\ & & \ddots & \ddots & \frac{n-1}{2} & \\ & & & \frac{n+2}{2} & x & n \\ & & & & \frac{n+1}{2} & x \end{pmatrix},$$

we immediately get the recurrences on a positive integer n,

$$\det H_{2n+1}\left(x\right) = \left(x-(2n+1)\right)\left(x+(2n+1)\right)\det H_{2n-1}\left(x\right), \quad \text{with } \det H_1\left(x\right) = x^2-1,$$

and

$$\det H_{2n}\left(x\right) = \left(x-2n\right)\left(x+2n\right)\det H_{2n-2}\left(x\right), \quad \text{with } \det H_0\left(x\right) = x,$$

which means that

$$\det H_{2n+1}\left(x\right) = \prod_{k=0}^{n}\left(x^2-(2k+1)^2\right)$$

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and

$$\det H_{2n}(x) = \prod_{k=0}^{n} \left(x^2 - (2k)^2\right).$$

3. The interlacing. Recently, it was proved in [12] the following:

THEOREM 3.1. *The eigenvalues of the matrix*

$$G_n = \begin{pmatrix} 0 & 1 & & & & & \\ 2n+2 & 0 & 2 & & & & \\ & 2n+1 & 0 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & n-1 & \\ & & & & n+4 & 0 & n \\ & & & & & n+3 & 0 \end{pmatrix}_{(n+1)\times(n+1)}$$

are

$$\left\{\pm 2\bar{k}\right\}_{\bar{k}=0}^{n},$$

with $\bar{k} \equiv n \pmod 2$. *That is, they are, for* G_{2n-1},

$$\{\pm 2, \pm 6, \pm 10, \ldots, \pm 2(2n-1)\},$$

while, for G_{2n}, *they are*

$$\{0, \pm 4, \pm 8, \pm 12, \ldots, \pm 4n\}.$$

Notice that Theorem 3.1 says that the eigenvalues of G_n are the double of the Sylvester-Kac matrix.

Suppose that $\hat{H}_n$ is the principal submatrix of order n obtained from H_n by the deletion of its last row and column. We find that $\hat{H}_n = \frac{1}{2}G_{n-1}$. Surprisingly, this means that H_n is a matrix with an n-Sylvester spectrum with principal submatrix $\hat{H}_n$ with an $(n-1)$-Sylvester spectrum. Therefore, the interlacing between the eigenvalues of H_n and $\hat{H}_n$ is:

$$-n < -n+1 < -n+2 < \cdots < -2 < -1 < 0 < 1 < 2 < \cdots < n-2 < n-1 < n.$$

4. The relation between the Sylvester-Kac matrix A_n and the new matrix H_n. Now we provide a similarity relation between the usual Sylvester-Kac matrix A_n and the tridiagonal matrix H_n. For this, define the upper triangular matrix $T = (T_{ij})$ of order n with the recursions for terms on the band entries for $0 \le r \le \lfloor (n-1)/2 \rfloor$,

$$\frac{T_{i,i+2r}}{T_{i+1,i+2r+1}} = \frac{i}{i+2r} \times \frac{2n-2r-i-1}{2(n-r-i)},$$

and for $0 \le i \le \lfloor (n-3)/2 \rfloor$,

$$\frac{T_{1,2i+1}}{T_{1,2i+3}} = \frac{2n-2i-3}{2i+1}$$

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and 0, otherwise, with the initial $T_{1,1} = L\left(n\right)$, where $L(n)$ is the leading coefficient of the Legendre polynomial $P(x)$, that is,

$$L(n) = \frac{(2n-1)!!}{n!},$$

where $(2n-1)!!$ is the double factorial defined by $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$.

Now, by straight computations, the inverse matrix $T^{-1} = (\Omega_{ij})$ is given by the recursion for the elements on the bands as shown for $0 \le r \le \lfloor (n-1)/2 \rfloor$

$$\frac{\Omega_{i,i+2r}}{\Omega_{i+1,i+2r+1}} = (-1)^r \, \frac{2i\,(n-i-r-1)\,(n-i)}{(i+2r)\,(n-i-1)\,(2n-i-1)}$$

and, for $0 \le i \le \lfloor (n-2)/2 \rfloor$,

$$\frac{\Omega_{1,2i+1}}{\Omega_{1,2i+3}} = \frac{2\,(n-i-2)}{2i+1}$$

and 0, otherwise, with the initial $\Omega_{1,1} = 1/L\left(n\right)$, where $L(n)$ is the leading coefficient of the Legendre polynomial $P(x)$, that is,

$$\Omega_{1,1} = \frac{n!}{(2n-1)!!}.$$

For instance, when $n = 10$, we have the matrix T and its inverse T^{-1} as follows

$$\begin{pmatrix}
\frac{12\,155}{128} & 0 & \frac{715}{128} & 0 & \frac{143}{128} & 0 & \frac{55}{128} & 0 & \frac{35}{128} & 0 \\
0 & \frac{12\,155}{128} & 0 & \frac{2145}{128} & 0 & \frac{715}{128} & 0 & \frac{385}{128} & 0 & \frac{315}{128} \\
0 & 0 & \frac{715}{8} & 0 & \frac{1001}{32} & 0 & \frac{495}{32} & 0 & \frac{175}{16} & 0 \\
0 & 0 & 0 & \frac{5005}{64} & 0 & \frac{715}{16} & 0 & \frac{1925}{64} & 0 & \frac{105}{4} \\
0 & 0 & 0 & 0 & \frac{1001}{16} & 0 & \frac{825}{16} & 0 & \frac{175}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{715}{16} & 0 & \frac{385}{8} & 0 & \frac{189}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{55}{2} & 0 & 35 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{55}{4} & 0 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}^{-1}$$

$$= \begin{pmatrix}
\frac{128}{12\,155} & 0 & -\frac{8}{12\,155} & 0 & \frac{12}{85\,085} & 0 & -\frac{1}{17\,017} & 0 & \frac{1}{24\,310} & 0 \\
0 & \frac{128}{12\,155} & 0 & -\frac{192}{85\,085} & 0 & \frac{16}{17\,017} & 0 & -\frac{8}{12\,155} & 0 & \frac{9}{12\,155} \\
0 & 0 & \frac{8}{715} & 0 & -\frac{4}{715} & 0 & \frac{3}{715} & 0 & -\frac{7}{1430} & 0 \\
0 & 0 & 0 & \frac{64}{5005} & 0 & -\frac{64}{5005} & 0 & \frac{12}{715} & 0 & -\frac{24}{715} \\
0 & 0 & 0 & 0 & \frac{16}{1001} & 0 & -\frac{30}{1001} & 0 & \frac{10}{143} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{16}{715} & 0 & -\frac{56}{715} & 0 & \frac{252}{715} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{55} & 0 & -\frac{14}{55} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{55} & 0 & -\frac{72}{55} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Now we can give our latest main result without proof which follows from matrix multiplication.

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THEOREM 4.1. *The similarity relation between the Sylvester-Kac matrix A_{n-1} and the matrix H_{n-1} can be given by*

$$A_{n-1} = TH_{n-1}T^{-1}.$$

For example, when $n = 9$, we have that

$$\begin{pmatrix} \frac{6435}{128} & 0 & \frac{429}{128} & 0 & \frac{99}{128} & 0 & \frac{45}{128} & 0 & \frac{35}{128} \\ 0 & \frac{6435}{128} & 0 & \frac{1287}{128} & 0 & \frac{495}{128} & 0 & \frac{315}{128} & 0 \\ 0 & 0 & \frac{3003}{64} & 0 & \frac{297}{16} & 0 & \frac{675}{64} & 0 & \frac{35}{4} \\ 0 & 0 & 0 & \frac{1287}{32} & 0 & \frac{825}{32} & 0 & \frac{315}{16} & 0 \\ 0 & 0 & 0 & 0 & \frac{495}{16} & 0 & \frac{225}{8} & 0 & \frac{105}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{165}{8} & 0 & \frac{189}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{45}{4} & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{15}{2} & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{13}{2} & 0 & \frac{5}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{11}{2} & 0 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{6435}{128} & 0 & \frac{429}{128} & 0 & \frac{99}{128} & 0 & \frac{45}{128} & 0 & \frac{35}{128} \\ 0 & \frac{6435}{128} & 0 & \frac{1287}{128} & 0 & \frac{495}{128} & 0 & \frac{315}{128} & 0 \\ 0 & 0 & \frac{3003}{64} & 0 & \frac{297}{16} & 0 & \frac{675}{64} & 0 & \frac{35}{4} \\ 0 & 0 & 0 & \frac{1287}{32} & 0 & \frac{825}{32} & 0 & \frac{315}{16} & 0 \\ 0 & 0 & 0 & 0 & \frac{495}{16} & 0 & \frac{225}{8} & 0 & \frac{105}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{165}{8} & 0 & \frac{189}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{45}{4} & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

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