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An observation on the determinant of a Sylvester-Kac type matrix

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Abstract

Based on a less-known result, we prove a recent conjecture concerning the determinant of a certain Sylvester-Kac type matrix related to some Lie Algebras. The determinant of an extension of that matrix is presented.

1 Introduction

Matrices and Lie algebras have an interesting long relation and share many problems. In a recent paper, Z. Hu and P.B. Zhang consider in [11] the polynomial

$$\det(z_0 I + z_1 A_1 + \cdots + z_s A_s) ,$$

where $A_1, \ldots, A_s$ are square matrices of the same order the I the identity matrix. Then they calculate the determinant of the finite dimensional irreducible representations of $sl(2, F)$, and show that is either zero or a product of some irreducible quadratic polynomials. In addition, it is proved that a finite dimensional Lie algebra is solvable if and only if the characteristic polynomial is completely reducible. For their purposes, they consider a specialised tridiagonal matrix with zero main diagonal, $(1, 2, \ldots, n)$ superdiagonal, and $(n, n-1, \ldots, 1)$ subdiagonal. Then they establish a conjecture, proved in two very particular cases.

The aim of this short note is to prove that conjecture based on a less-known result by W. Chu in [3]. We also provide a new general formula containing other particular known determinants. This formula can be used to extend [11], and useful both in Lie algebras and matrix theory.

Key Words: Sylvester-Kac matrix, Clement matrix, determinant, eigenvalues, Lie algebras.
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2 The conjecture

Quite recently, in order to find formulas for the determinants of some Lie algebras, Z. Hu and P.B. Zhang proposed in [11] the following conjecture.

Conjecture 1. *The determinant of the matrix* $(n+1)\times(n+1)$

$$J_n(z_0,z_1)=\begin{pmatrix} z_0+nz_1 & 1 & & & & \\ n & z_0+(n-2)z_1 & 2 & & & \\ & n-1 & z_0+(n-4)z_1 & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & z_0-(n-2)z_1 & n \\ & & & & 1 & z_0-nz_1 \end{pmatrix}$$

is

$$\prod_{k=0}^{n}\left(z_0-(n-2k)\sqrt{z_1^2+1}\right).$$

Notice that Conjecture 1 is equivalent to state that the eigenvalues of $J_n(0,z_1)$ are

$$\pm(n-2k)\sqrt{z_1^2+1}\,,\quad \text{for } k=0,1,\dots,\lfloor n/2\rfloor.$$

The matrix $J_n(z_0,z_1)$ can be easily identified as an extension of the so-called Sylvester-Kac matrix. In fact, setting $z_1=0$ we find the characteristic matrix of the Sylvester-Kac matrix, also known as Clement matrix,

$$\begin{pmatrix} 0 & 1 & & & \\ n & 0 & 2 & & \\ & n-1 & 0 & \ddots & \\ & & \ddots & \ddots & n \\ & & & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix (that is, $\det J_n(x,0)$) was first conjecture in [20], by the 19th century British mathematician James Joseph Sylvester celebrated, among other facets, as the founder of the American Journal of Mathematics, in 1878.

A fully comprehensive list of results on the different proofs for Sylvester's conjecture and the eigenpairs of non-trivial extensions of the Sylvester-Kac matrix can be found in [1–10, 12–19, 21, 22].

The aim of this short note is to prove Conjecture 1 based on a result by W. Chu in [3]. We also provide a general result containing other particular known determinants.

3 An extension to the Sylvester-Kac matrix

In 2010, cleverly based on two generalized Fibonacci sequences, W. Chu proved the following theorem.

Theorem 3.1 ([3]). *The determinant of the matrix* $(n+1)\times(n+1)$

$$M_n(x,y,u,v)=\begin{pmatrix} x & u & & & & \\ nv & x+y & 2u & & & \\ & (n-1)v & x+2y & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2v & x-(n-1)y & nu \\ & & & & v & x+ny \end{pmatrix}$$

is

$$\prod_{k=0}^{n}\left(x+\frac{ny}{2}+\frac{n-2k}{2}\sqrt{y^2+4uv}\right).$$

Of course, the formula for the determinant in Theorem 3.1 can be rewritten as

$$\prod_{k=0}^{\lfloor n/2\rfloor}\left(\left(x+\frac{ny}{2}\right)^2-\frac{(n-2k)^2}{4}(y^2+4uv)\right).$$

Now setting $x=z_0+nz_1$, $y=-2z_1$, and $u=v=1$, we prove immediately Conjecture 1.

Moreover, in the spirit of [1,9,10], using Theorem 3.1, we can also conclude the following theorem.

Theorem 3.2. *The eigenvalues of*

$$M_n^{\pm}(a,b,r)=\begin{pmatrix} nar & b & & & & \\ na & ((n-1)a\pm b)r & 2b & & & \\ & (n-1)a & ((n-2)a\pm 2b)r & 3b & & \\ & & (n-2)a & \ddots & \ddots & \\ & & & \ddots & \ddots & nb \\ & & & & a & \pm nbr \end{pmatrix}$$

are

$$\frac{1}{2}\left(nr(a\pm b)+(n-2k)\sqrt{4ab+r^2(a\mp b)^2}\right),$$

for $k=0,1,\ldots,n$.

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