ARS MATHEMATICA CONTEMPORANEA

ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 19 (2020) 25–35
https://doi.org/10.26493/1855-3974.2028.cb4
(Also available at http://amc-journal.eu)

Results on the domination number and the total domination number of Lucas cubes*

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Received 27 June 2019, accepted 2 July 2020, published online 28 October 2020

Abstract

Lucas cubes are special subgraphs of Fibonacci cubes. For small dimensions, their domination numbers are obtained by direct search or integer linear programming. For larger dimensions some bounds on these numbers are given. In this work, we present the exact values of total domination number of small dimensional Lucas cubes and present optimization problems obtained from the degree information of Lucas cubes, whose solutions give better lower bounds on the domination numbers and total domination numbers of Lucas cubes.

Keywords: Lucas cube, Fibonacci cube, domination number, total domination number, integer linear programming.

Math. Subj. Class. (2020): 05C69, 68R10, 11B39

1 Introduction

Fibonacci cubes and Lucas cubes are special subgraphs of the hypercube graph, which are introduced as an alternative model for interconnection networks [6, 11]. The structural and enumerative properties of these graphs are considered from various point of view in the literature [2, 6, 7, 8, 9, 10, 11, 15].

Let Q_n denote the hypercube of dimension $n \geq 1$. It is the graph with vertex set represented by all binary strings of length n and two vertices in Q_n are adjacent if they differ in one coordinate. For convenience $Q_0 = K_1$. *Fibonacci strings* of length n are defined as the binary strings $b_1 b_2 \ldots b_n$ such that $b_i \cdot b_{i+1} = 0$ for all $i = 0, 1, \ldots, n-1$, that is, binary strings of length n not containing two consecutive 1s. Using this representation n dimensional *Fibonacci cube* Γ_n is defined as the subgraph of Q_n induced by the vertices

*The author is grateful to the anonymous reviewers for their valuable comments and careful reading and would like to thank Ömer Eğecioğlu and Elif Saygı for useful discussions and suggestions.

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whose string representations are Fibonacci strings. *Lucas strings* of length n are defined as the Fibonacci strings $b_1 b_2 \dots b_n$ such that $b_1 \cdot b_n = 0$. Similar to the Fibonacci cubes n dimensional *Lucas cube* Λ_n is defined as the subgraph of Γ_n induced by the vertices whose string representations are Lucas strings.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. $D \subseteq V$ is called a *dominating set* of G if every vertex in V either belongs to D or is adjacent to some vertex in D. Then the *domination number* $\gamma(G)$ of G is defined as the minimum cardinality of a dominating set of the graph G. Similarly, $D \subseteq V$ is called a *total dominating set* of a graph G without isolated vertex if every vertex in V is adjacent to some vertex in D and the *total domination number* $\gamma_t(G)$ of G is defined as the minimum cardinality of a total dominating set of G. The domination numbers of Γ_n and Λ_n are first considered in [2, 12]. Using integer linear programming, domination numbers of Γ_n and Λ_n are considered in [7] and total domination number of Γ_n is considered in [1]. Furthermore, upper bounds and lower bounds on $\gamma(\Gamma_n), \gamma_t(\Gamma_n), \gamma(\Lambda_n)$ are obtained in [1, 2, 13] and they are improved for Γ_n in [14].

In this work, we present optimization problems obtained from the degree information of Lucas cubes, whose solutions give better lower bounds on the domination numbers and total domination numbers of Lucas cubes. Our aim is to improve the known results on $\gamma(\Lambda_n)$ and present new results on $\gamma_t(\Lambda_n)$. Furthermore, we introduce the up-down degree polynomials for Λ_n containing the degree information of all vertices $V(\Lambda_n)$ in more detail. Using these polynomials we define optimization problems whose solutions give lower bound on $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$.

2 Preliminaries

For $n \geq 2$ we will use the fundamental decomposition of Γ_n (see, [8]):

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}, \tag{2.1}$$

where $\Gamma_0 = Q_0$ and $\Gamma_1 = Q_1$. Here note that $0\Gamma_{n-1}$ is the subgraph of Γ_n induced by the vertices that start with 0 and Γ_{n-2} is the subgraph of Γ_n induced by the vertices that start with 10. Furthermore, $0\Gamma_{n-1}$ has a subgraph isomorphic to $00\Gamma_{n-2}$, and there exists a perfect matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$. Similar to this decomposition for $n \geq 3$ Lucas cubes can be written as

$$\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0, \tag{2.2}$$

where $10\Gamma_{n-3}0$ is the subgraph of Λ_n induced by the vertices that start with 10 and end with 0. Here, there exists a perfect matching between $10\Gamma_{n-3}0$ and $00\Gamma_{n-3}0 \subset 0\Gamma_{n-1}$. By convention, $\Lambda_1 = \Gamma_0$ and $\Lambda_2 = \Gamma_2$.

The number of vertices of the Γ_n is f_{n+2}, where f_n are the Fibonacci numbers defined as $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Similarly, the number of vertices of the Λ_n is L_n, where L_n are the Lucas numbers defined as $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Let x, y be two binary strings of length n. Then the *Hamming distance* between x and y, $d_H(x, y)$ is the number of coordinates in which they differ. The *Hamming weight* of x, $w(x)$ is the number of nonzero coordinates in x. Note that Hamming distance is the usual graph distance in Q_n.

In Figure 1 we present small dimensional Lucas cubes and a minimal total dominating set with circled vertices for $2 \leq n \leq 5$.

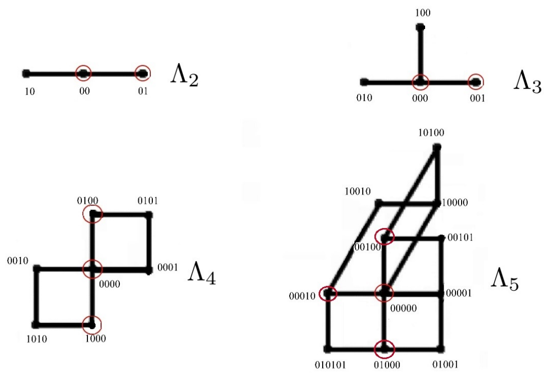

Figure 1: Lucas cubes and their minimal total dominating sets for $2 \leq n \leq 5$.

3 Integer linear programming for domination numbers

In this section, we describe a linear programming problem used in [7] for finding the domination number of Γ_n and Λ_n. A similar approach is used in [1] for finding the total domination number of Γ_n. The main difficulty for these methods are the number of variables and the number of constraints which are equal to the number of vertices in Γ_n and Λ_n. Using this approach we obtain the total domination number of Λ_n for $n \leq 12$.

Let $N(v)$ denote the set of vertices adjacent to v and $N[v] = N(v) \cup \{v\}$. Suppose each vertex $v \in V(\Lambda_n)$ is associated with a binary variable x_v. The problems of determining $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$ can be expressed as a problem of minimizing the objective function

$$\sum_{v \in V(\Lambda_n)} x_v \tag{3.1}$$

subject to the following constraints for every $v \in V(\Lambda_n)$:

$$\sum_{a \in N[v]} x_a \geq 1 \quad \text{(for domination number)},$$

$$\sum_{a \in N(v)} x_a \geq 1 \quad \text{(for total domination number)}.$$

The value of the objective function gives $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$ respectively. Note that this problem has L_n variables and L_n constraints. In [7] $\gamma(\Lambda_n)$ is obtained up to $n = 11$ and for larger values of n as the number of variables increases no results are presented.

We implemented the integer linear programming problem (3.1) using CPLEX in NEOS Server [3, 4, 5] for $n \leq 12$ and obtain the values of $\gamma_t(\Lambda_n)$ for $n \leq 12$ and obtain the estimates $49 \leq \gamma(\Lambda_{12}) \leq 54$ (takes approximately 2 hours). We collect the known values of $\gamma(\Lambda_n)$ for $n \leq 11$ (see, [7]) and the new values of $\gamma_t(\Lambda_n)$ that we obtained from (3.1) for $n \leq 12$ in Table 1.

The fundamental decompositions (2.1) and (2.2) of Γ_n and Λ_n are used to obtain the following relations between $\gamma(\Lambda_n)$ and $\gamma(\Gamma_n)$. The main idea in the proof is to partition the set of vertices into disjoint subsets.

Table 1: Values of $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$ for $n \leq 12$.

n	2	3	4	5	6	7	8	9	10	11	12
$\lvert V(\Lambda_n)\rvert$	3	4	7	11	18	29	47	76	123	199	322
$\gamma(\Lambda_n)$	1	1	3	4	5	7	11	16	23	35	
$\gamma_t(\Lambda_n)$	2	2	3	4	7	9	13	19	27	41	58

Proposition 3.1 ([2, Proposition 3.1]). *Let $n \geq 4$, then*

(i) $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-3})$,

(ii) $\gamma(\Lambda_n) \leq \gamma(\Gamma_n) \leq \gamma(\Lambda_n) + \gamma(\Gamma_{n-4})$.

Using a similar idea we obtain the following result.

Proposition 3.2. *Let $n \geq 4$, then*

(i) $\gamma_t(\Lambda_n) \leq \gamma_t(\Gamma_{n-1}) + \gamma_t(\Gamma_{n-3})$,

(ii) $\gamma_t(\Lambda_n) \leq \gamma_t(\Gamma_n) \leq \gamma_t(\Lambda_n) + \gamma_t(\Gamma_{n-4})$.

Proof. The proof mimics the proof of [2, Proposition 3.1].

(i): The vertices of Λ_n can be partitioned into vertices that start with 0 and vertices that start with 1. The subgraphs induced by these vertices are isomorphic to Γ_{n-1} and Γ_{n-3} respectively, hence we infer that $\gamma_t(\Lambda_n) \leq \gamma_t(\Gamma_{n-1}) + \gamma_t(\Gamma_{n-3})$.

(ii): Let D_T be a minimal total dominating set of Γ_n and set

$$D'_T = \{\alpha \mid \alpha \text{ is a Lucas string from } D_T\} \cup \{0\alpha0 \mid 1\alpha1 \in D_T\}.$$

Note that $|D'_T| \leq |D_T|$ and a vertex of the form $1\alpha1$ dominates two Lucas vertices of the form $0\alpha1$ and $1\alpha0$. Since these two vertices are dominated by $0\alpha0$, we say that D'_T is a dominating set of Λ_n. Then we need to show that it is also a total dominating set. We know that every vertex in $v \in V(\Lambda_n) \subseteq V(\Gamma_n)$ is adjacent to some vertex $\beta \in D_T$. Then if $\beta \in D'_T$ we are done. Otherwise, β must be of the form $1\alpha1 \in D_T$. In this case $v \in V(\Lambda_n)$ must be of the form $1\alpha0$ or $0\alpha1$, which means that v is also adjacent to a vertex of the form $0\alpha0 \in D'_T$. It follows that $\gamma_t(\Lambda_n) \leq \gamma_t(\Gamma_n)$. On the other hand, a total dominating set of Λ_n dominates all vertices of Γ_n but the vertices of the form $10\alpha01$ where the subgraph induced by these vertices is isomorphic to Γ_{n-4}. Hence we have $\gamma_t(\Gamma_n) \leq \gamma_t(\Lambda_n) + \gamma_t(\Gamma_{n-4})$. □

Considering the vertices of high degrees a lower bound on $\gamma(\Lambda_n)$ is obtained in [2, Theorem 3.5] as $\gamma(\Lambda_n) \geq \lceil \frac{L_n - 2n}{n-3} \rceil$ where $n \geq 7$. Combining this result with the fact that $\gamma_t(\Lambda_n) \geq \gamma(\Lambda_n)$ we have the following lower bound on $\gamma_t(\Lambda_n)$.

Proposition 3.3. *For any $n \geq 7$, we have*

$$\gamma_t(\Lambda_n) \geq \gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n-3} \right\rceil.$$

4 Up-down degree enumerator polynomial

In this section we present the up-down degree enumerator polynomial for Λ_n similar to the one for Γ_n given in [14]. Using this polynomial we write optimization problems whose solutions give lower bounds on $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$.

By the definition of the edge set $E(\Lambda_n)$, $(v, v') \in E(\Lambda_n)$ if and only if the number of different coordinates of v and v' is 1, that is, the Hamming distance $d_H(v, v') = 1$. Here we have at most two kinds of neighbor v' for a vertex in $v \in V(\Lambda_n)$, whose weights can take the values $w(v) \pm 1$. If $w(v') = w(v) + 1$ we call v' is an *up neighbor* of v and if $w(v') = w(v) - 1$ we call v' is a *down neighbor* of v. We denote the number of up neighbors of v by u and the number of down neighbors of v by d which is equal to the $w(v)$ by the definition of Λ_n. Note that if the degree of v is k then we have $u = k - d$. For each fixed $v \in V(\Lambda_n)$ having degree $k = \deg(v)$, we write a monomial $x^u y^d$ where $d = w(v)$ is the Hamming weight of v and u is $k - d$. We call the polynomial

$$P_{\Lambda_n}(x, y) = \sum_{v \in V(\Lambda_n)} x^{\deg(v) - w(v)} y^{w(v)} = \sum_{v \in V(\Lambda_n)} x^u y^d$$

as the *up-down degree enumerator polynomial* of Λ_n.

We need the following useful result given in [10] to obtain the recursive structure of $P_{\Lambda_n}(x, y)$. Let $\ell_{n,k,w}$ be the number of vertices in Λ_n of degree k and weight w.

Theorem 4.1 ([10, Theorem 5.2])**.** *For all n, k, w such that $n \geq 2$, $1 \leq k \leq n$ and $0 \leq w \leq n$,*

$$\ell_{n,k,w} = \binom{w-1}{2w+k-n}\binom{n-2w}{k-w} + 2\binom{w}{2w+k-n}\binom{n-2w-1}{k-w}.$$

Let $\ell'_{n,u,d}$ be the number of vertices in Λ_n whose number of up neighbors are u and number of down neighbors are d. Setting $k = u + d$ and $w = d$ in Theorem 4.1 we have

$$\ell'_{n,u,d} = \binom{d-1}{3d+u-n}\binom{n-2d}{u} + 2\binom{d}{3d+u-n}\binom{n-2d-1}{u}. \tag{4.1}$$

Then using (4.1) we can write the up-down degree enumerator polynomial of Λ_n as

$$P_{\Lambda_n}(x, y) = \sum_{u,d} \ell'_{n,u,d}\, x^u y^d, \tag{4.2}$$

where $0 \leq u, d \leq n$. Furthermore, using (4.1) and (4.2) we obtain the following recursive relation which is very useful to calculate $P_{\Lambda_n}(x, y)$.

Theorem 4.2. *Let $P_{\Lambda_n}(x, y)$ be the up-down degree enumerator polynomial of Λ_n. Then for $n \geq 5$ we have*

$$P_{\Lambda_n}(x, y) = xP_{\Lambda_{n-1}}(x, y) + yP_{\Lambda_{n-2}}(x, y) + (y - xy)P_{\Lambda_{n-3}}(x, y), \tag{4.3}$$

where

$$P_{\Lambda_2}(x, y) = x^2 + 2y, \quad P_{\Lambda_3}(x, y) = x^3 + 3y \quad \textit{and} \quad P_{\Lambda_4}(x, y) = x^4 + 4xy + 2y^2.$$

Proof. The initial conditions are clear from the definition of Λ_n. For fixed integers $1 \leq u < n$ and $2 \leq d < \lfloor \frac{n}{2} \rfloor$, the coefficient of the monomial $x^u y^d$ in the right hand side of the equation (4.3) is the sum of $\ell'_{n-1,u-1,d}$ coming from $xP_{\Lambda_{n-1}}(x,y)$, $\ell'_{n-2,u,d-1}$ coming from $yP_{\Lambda_{n-2}}(x,y)$, $\ell'_{n-3,u,d-1}$ coming from $yP_{\Lambda_{n-3}}(x,y)$ and $-\ell'_{n-3,u-1,d-1}$ coming from $-xyP_{\Lambda_{n-3}}(x,y)$. Then we need to show that

$$\ell'_{n,u,d} = \ell'_{n-1,u-1,d} + \ell'_{n-2,u,d-1} + \ell'_{n-3,u,d-1} - \ell'_{n-3,u-1,d-1}.$$

By setting $X = 3d + u - n$ and $Y = n - 2d$ in (4.1) and using the binomial identities

$$\binom{m}{k} = \frac{m}{k}\binom{m-1}{k-1} = \frac{m+1-k}{k}\binom{m}{k-1} = \frac{m}{m-k}\binom{m-1}{k}$$

we have

$$\begin{aligned}
&\ell'_{n-1,u-1,d} + \ell'_{n-2,u,d-1} + \ell'_{n-3,u,d-1} - \ell'_{n-3,u-1,d-1} \\
&\quad = \binom{d-1}{X}\binom{Y-1}{u-1} + 2\binom{d}{X}\binom{Y-2}{u-1} \\
&\qquad + \binom{d-2}{X-1}\binom{Y}{u} + 2\binom{d-1}{X-1}\binom{Y-1}{u} \\
&\qquad + \binom{d-2}{X}\binom{Y-1}{u} + 2\binom{d-1}{X}\binom{Y-2}{u} \\
&\qquad - \binom{d-2}{X-1}\binom{Y-1}{u-1} + 2\binom{d-1}{X-1}\binom{Y-2}{u-1} \\
&\quad = \binom{d-1}{X}\binom{Y}{u}\left[\frac{u}{Y} + \frac{X}{d-1} + \frac{d-1-X}{d-1}\cdot\frac{Y-u}{Y} - \frac{X}{d-1}\cdot\frac{u}{Y}\right] \\
&\qquad + 2\binom{d}{X}\binom{Y-1}{u}\left[\frac{u}{Y-1} + \frac{X}{d} + \frac{d-X}{d}\cdot\frac{Y-1-u}{Y-1} - \frac{X}{d}\cdot\frac{u}{Y-1}\right] \\
&\quad = \binom{d-1}{X}\binom{Y}{u} + 2\binom{d}{X}\binom{Y-1}{u} \\
&\quad = \ell'_{n,u,d}.
\end{aligned}$$

In particular, the case $d = 0$ corresponds to the all 0 vertex in Λ_n and we have $\ell'_{n,n,0} = 1$, which means that the coefficient of the terms $x^n y^0$ in both sides of (4.3) are 1. Similarly, the case $u = 0$ corresponds to the vertices in Λ_n whose weights are $\lfloor \frac{n}{2} \rfloor$ and we have (see, Remark 5.1)

$$\ell'_{n,0,\lfloor \frac{n}{2} \rfloor} = \begin{cases} n & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Then one can easily see that the coefficient of the terms $x^0 y^{\lfloor \frac{n}{2} \rfloor}$ in both sides of (4.3) are equal to each other. The only remaining particular case is $d = 1$. For $P_{\Lambda_n}(x,y)$ this case corresponds to the vertices in Λ_n whose weights are 1. We know that there are n such vertices in Λ_n and their number of up neighbors are $n-3$. That is, the coefficient of the term $x^{n-3}y$ in $P_{\Lambda_n}(x,y)$ is n. On the other hand the coefficient of the term $x^{n-3}y$ is $n-1$ in $xP_{\Lambda_{n-1}}(x,y)$; 0 in $yP_{\Lambda_{n-2}}(x,y)$ and 1 in $(y-xy)P_{\Lambda_{n-3}}(x,y)$ respectively. Hence the coefficient of the terms $x^u y^d$ in both sides of (4.3) are equal to each other for all cases. □

Remark 4.3. The recursive relation for the up-down degree enumerator polynomial of Λ_n in Theorem 4.2 is the same with the recursive relation for the up-down degree enumerator polynomial of Γ_n, which is proved using the fundamental decomposition $\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}$. The only differences are the initial polynomials. For the proof we directly used the degree information of Λ_n obtained in [10], since Λ_n do not have a decomposition like $0\Lambda_{n-1} + 10\Lambda_{n-2}$.

5 Lower bounds on domination numbers using optimization problems

In this section, we present optimization problems giving lower bounds on $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$, whose number of variables and number of constraints are fewer than the general optimization problem described in Section 3.

We use the up-down degree enumerator polynomial $P_{\Lambda_n}(x, y)$ to construct an optimization problem, which is similar to the optimization problem given in [14]. Let D and D_T be a dominating set and a total dominating set of Λ_n respectively. Let $v_D \in D$ $\big(v_D \in D_T$ respectively$\big)$ and $x^u y^d$ be its corresponding monomial in $P_{\Lambda_n}(x, y)$. Then v_D dominates u distinct vertices $v \in V(\Lambda_n)$ having weight $w(v) = w(v_D) + 1$ and d distinct vertices $v \in V(\Lambda_n)$ having weight $w(v) = w(v_D) - 1$. Note that for all $v_D \in D$ $\big(v_D \in D_T$ respectively$\big)$ some of the vertices of Λ_n may be dominated more than one times. Note that for every vertex $v \in V(\Lambda_n)$ there must exist at least one vertex $v_D \in N[v] \cap D_T$ with $w(v_D) = w(v) \mp 1$ or $v_D = v$ for the dominating set D and $v_D \in N(v) \cap D_T$ with $w(v_D) = w(v) \mp 1$ for the total dominating set D_T.

Now we write the up-down degree enumerator polynomial of Λ_n (see, 4.2) as

$$P_{\Lambda_n}(x, y) = \sum_{u,d} c_d^u x^u y^d, \tag{5.1}$$

where $c_d^u = \ell'_{n,u,d}$. For each pair (u, d) in the monomials of the up-down degree enumerator polynomial $P_{\Lambda_n}(x, y)$ we associate an integer variable z_d^u which counts the number of vertices in D or D_T having d down neighbors and u up neighbors. For any fixed value of d, the number of vertices having weight d gives the bounds $0 \leq z_d^u \leq c_d^u$. Our aim is to minimize $|D|$ for domination number and to minimize $|D_T|$ for total domination number. Hence our objective function is to minimize

$$\sum_{u,d} z_d^u.$$

To dominate all the vertices having a fixed weight d such that $1 \leq d \leq \lfloor \frac{n}{2} \rfloor - 1$ we must have the following constraints r_d for domination number and r'_d for the total domination number.

$$r_d: \quad \sum_u \left(u \cdot z_{d-1}^u + z_d^u + (d+1) \cdot z_{d+1}^u\right) \geq \sum_u c_d^u$$

$$r'_d: \quad \sum_u \left(u \cdot z_{d-1}^u + (d+1) \cdot z_{d+1}^u\right) \geq \sum_u c_d^u$$

since any vertex corresponding to the monomial $x^u y^{d-1}$ can dominate u distinct vertices (u up neighbors) having weight d and any vertex corresponding to the monomial $x^{u'} y^{d+1}$

can dominate $d+1$ distinct vertices ($d+1$ down neighbors) having weight d. By the same argument, for $d=0$ we must have

$$r_0\colon \quad \sum_u z_0^u + z_1^u \geq \sum_u c_0^u = 1 \qquad \text{and} \qquad r_0'\colon \quad \sum_u z_1^u \geq \sum_u c_0^u$$

and for $d = \lfloor \frac{n}{2} \rfloor$ we must have

$$r_{\lfloor \frac{n}{2} \rfloor}\colon \qquad \sum_u u \cdot z^u_{\lfloor \frac{n}{2} \rfloor - 1} + z^u_{\lfloor \frac{n}{2} \rfloor} \geq \sum_u c^u_{\lfloor \frac{n}{2} \rfloor} = \begin{cases} n & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

$$r'_{\lfloor \frac{n}{2} \rfloor}\colon \qquad \sum_u u \cdot z^u_{\lfloor \frac{n}{2} \rfloor - 1} \geq \sum_u c^u_{\lfloor \frac{n}{2} \rfloor} = \begin{cases} n & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Now subject to the above constraints $r_0, \dots, r_{\lfloor \frac{n}{2} \rfloor}$ (constraints $r_0', \dots, r'_{\lfloor \frac{n}{2} \rfloor}$) the value of the objective function will be a lower bound on $\gamma(\Lambda_n)$ ($\gamma_t(\Lambda_n)$, respectively).

Remark 5.1. The number of vertices of Λ_n having weight d is equal to the right hand side of the above constraints r_d and r_d'. By setting $k = u + d$ and $w = d$ in [10, Corollary 5.3] we have

$$\sum_u c_d^u = \sum_{u=0}^{n-d} \ell'_{n,u,d} = \binom{n-d}{d} + \binom{n-d-1}{n-2d}.$$

Remark 5.2. The number of variables z_d^u in our optimization problem is equal to the number of monomials in $P_{\Lambda_n}(x,y)$. Assume that n is even. By the string representation of the vertices in Λ_n we have $n-3d \leq u \leq n-2d-1$. The bounds come from the maximum number of the sub-strings 010 and 10 in the representation of the vertices. That is, u can take $n-2d-1-(n-3d)+1 = d$ distinct values when d ranges from 1 up to $\lfloor \frac{n}{3} \rfloor$ and u can take $n-2d$ distinct values when $\frac{n}{3}+1 \leq d < \lfloor \frac{n}{2} \rfloor$. Furthermore, u can take only one values for $d=0$ and $d = \lfloor \frac{n}{2} \rfloor$. Therefore, the number of variables z_d^u becomes

$$2 + \sum_{d=1}^{\lfloor \frac{n}{3} \rfloor} d + \sum_{d=\lfloor \frac{n}{3} \rfloor + 1}^{\lfloor \frac{n}{2} \rfloor - 1} (n-2d)$$

which is equal to

$$2 + \frac{3}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 - \frac{2n}{3} \right) + \left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) - n. \tag{5.2}$$

For $n \geq 2$ this sequence starts as $2, 2, 3, 3, 5, 5, 7, 8, 10, 11, 14, 15, 18, 20, 23, 25, 29, \dots$ Note that in (3.1) the number of variables is L_n, which exhibit exponential growth. In our case, if we omit the floor functions in (5.2) then the number of variables z_d^u is approximately equals to $2 + \frac{n^2}{12}$.

For $n = 12$ we illustrate our optimization problem as follows. First we obtain $P_{\Lambda_{12}}(x,y)$

by using the recursion in Theorem 4.2 as

$$
\begin{aligned}
P_{\Lambda_{12}}(x,y) = {} & 2y^6 + \\
& 12y^5x + 24y^5 + \\
& 12y^4x^3 + 54y^4x^2 + 36y^4x + 3y^4 + \\
& 12y^3x^5 + 60y^3x^4 + 40y^3x^3 + \\
& 12y^2x^7 + 42y^2x^6 + \\
& 12yx^9 + \\
& x^{12}
\end{aligned}
$$

Then using $P_{\Lambda_{12}}(x,y)$ we have the corresponding optimization problem:

Objective function:

$$
minimize: \; z_0^{12} + z_1^9 + z_2^7 + z_2^6 + z_3^5 + z_3^4 + z_3^3 + z_4^3 + z_4^2 + z_4^1 + z_4^0 + z_5^1 + z_5^0 + z_6^0;
$$

Constraints for $\gamma(\Lambda_{12})$:

$$
\begin{aligned}
r_6: & \quad z_5^1 + z_6^0 \geq 2; \\
r_5: & \quad 3z_4^3 + 2z_4^2 + z_4^1 + z_5^1 + z_5^0 + 6z_6^0 \geq 36; \\
r_4: & \quad 5z_3^5 + 4z_3^4 + 3z_3^3 + z_4^3 + z_4^2 + z_4^1 + z_4^0 + 5z_5^1 + 5z_5^0 \geq 105; \\
r_3: & \quad 7z_2^7 + 6z_2^6 + z_3^5 + z_3^4 + z_3^2 + 4z_4^3 + 4z_4^2 + 4z_4^1 + 4z_4^0 \geq 112; \\
r_2: & \quad 9z_1^9 + z_2^7 + z_2^6 + 3z_3^5 + 3z_3^4 + 3z_3^3 \geq 54; \\
r_1: & \quad 12z_0^{12} + z_1^9 + 2z_2^7 + 2z_2^6 \geq 12; \\
r_0: & \quad z_0^{12} + z_1^9 \geq 1;
\end{aligned}
$$

Constraints for $\gamma_t(\Lambda_{12})$:

$$
\begin{aligned}
r'_6: & \quad z_5^1 \geq 2; \\
r'_5: & \quad 3z_4^3 + 2z_4^2 + z_4^1 + 6z_6^0 \geq 36; \\
r'_4: & \quad 5z_3^5 + 4z_3^4 + 3z_3^3 + 5z_5^1 + 5z_5^0 \geq 105; \\
r'_3: & \quad 7z_2^7 + 6z_2^6 + 4z_4^3 + 4z_4^2 + 4z_4^1 + 4z_4^0 \geq 112; \\
r'_2: & \quad 9z_1^9 + 3z_3^5 + 3z_3^4 + 3z_3^3 \geq 54; \\
r'_1: & \quad 12z_0^{12} + 2z_2^7 + 2z_2^6 \geq 12; \\
r'_0: & \quad z_1^9 \geq 1;
\end{aligned}
$$

Bounds:

$$
\begin{array}{ccccccc}
z_0^{12} \leq 1; & z_1^9 \leq 12; & z_2^7 \leq 12; & z_2^6 \leq 42; & z_3^5 \leq 12; & z_3^4 \leq 60; & z_3^3 \leq 40; \\
z_4^3 \leq 12; & z_4^2 \leq 54; & z_4^1 \leq 36; & z_4^0 \leq 3; & z_5^1 \leq 12; & z_5^0 \leq 24; & z_6^0 \leq 2.
\end{array}
$$

Depending on the constraints r_d and r'_d $(d = 0, 1, \ldots, 6)$ the value of the objective function gives a lower bound on $\gamma(\Lambda_{12})$ and $\gamma_t(\Lambda_{12})$ respectively. The above problem has only 14 variables and 7 constraints (instead of having $L_{12} = 322$ variables and 322 constraints as in (3.1)). To find lower bounds on $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$ one can use the up-down degree enumerator polynomial $P_{\Lambda_n}(x, y)$ of Λ_n in Theorem 4.2 and one can write an optimization problem having fewer number of variables z_d^u (see Remark 5.2) and $\lfloor \frac{n}{2} \rfloor + 1$ constraints r_d or r'_d. The solutions of the optimization problems give lower bounds on $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$. It is easy to see that the number of variables and the number of constraints in our optimization problems are very smaller than the ones in the optimization problem (3.1).

For illustration we implemented the above integer linear programming problem using CPLEX in NEOS Server [3, 4, 5] for $12 \le n \le 26$ and immediately (less than 0.02 seconds) obtain the lower bounds on $\gamma(\Lambda_n)$ and $\gamma_t(\Lambda_n)$ presented in Table 2 and Table 3 (better than the ones in Proposition 3.3). Note that for $n = 26$, the number of variables in our optimization problem is 58 by Remark 5.2 and the number of constraints is 14, on the other hand, these numbers are equal to $L_{26} = 271443$ for the general optimization problem (3.1). In addition, the upper bounds in these tables are obtained by Proposition 3.1 and Proposition 3.2 by using the upper bounds on the values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ given in [14] for $n \ge 14$.

Table 2: Current best bounds on $\gamma(\Lambda_n)$, $12 \le n \le 26$.

n	$\gamma(\Lambda_n)$	n	$\gamma(\Lambda_n)$	n	$\gamma(\Lambda_n)$
12	$49^* - 54$	17	$310 - 555$	22	$2686 - 6140$
13	$61^* - 86$	18	$471 - 895$	23	$4184 - 9935$
14	$89 - 132$	19	$725 - 1450$	24	$6519 - 16075$
15	$134 - 215$	20	$1114 - 2345$	25	$10163 - 26010$
16	$203 - 340$	21	$1724 - 3795$	26	$15835 - 42085$

Table 3: Current best bounds on $\gamma_t(\Lambda_n)$, $12 \le n \le 26$.

n	$\gamma_t(\Lambda_n)$	n	$\gamma_t(\Lambda_n)$	n	$\gamma_t(\Lambda_n)$
12	$\mathbf{58}^*$	17	$340 - 567$	22	$2893 - 6140$
13	$77^* - 95$	18	$514 - 909$	23	$4490 - 9935$
14	$101 - 145$	19	$787 - 1450$	24	$6974 - 16075$
15	$151 - 231$	20	$1205 - 2345$	25	$10839 - 26010$
16	$225 - 362$	21	$1862 - 3795$	26	$16838 - 42085$

Remark 5.3. It is shown in [1, 7] that $\gamma(\Gamma_9) = 17$, $\gamma(\Gamma_{10}) = 25$, $54 \le \gamma(\Gamma_{12}) \le 61$ and $78 \le \gamma(\Gamma_{13}) \le 93$ (shown in [14]). Substituting these results in Proposition 3.1 we obtain the bounds for $n = 13$ in Table 2.

Similarly, it is shown in [1, 7] that $\gamma_t(\Gamma_9) = 20$, $\gamma_t(\Gamma_{10}) = 30$, $\gamma_t(\Gamma_{12}) = 65$ and $97 \le \gamma_t(\Gamma_{13}) \le 101$. Substituting these results in Proposition 3.2 we obtain the bounds for $n = 13$ in Table 3.

Note that our optimization problems obtained from up-down degree enumerator polynomial give $\gamma(\Lambda_{12}) \ge 39$, $\gamma_t(\Lambda_{12}) \ge 45$ and $\gamma(\Lambda_{13}) \ge 59$, $\gamma_t(\Lambda_{13}) \ge 68$. Furthermore, using (3.1) we obtain that $49 \leq \gamma(\Lambda_{12}) \leq 54$ and $\gamma_t(\Lambda_{12}) = 58$. For these reasons we put a * to the lower bounds for the cases $n = 12$ and $n = 13$ in Table 2 and Table 3.

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