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Rates of A-statistical convergence of operators in the space of locally integrable functions

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Abstract

Motivated by our earlier work on the statistical approximation of locally integrable functions by positive linear operators, we study rates of A-statistical convergence of a sequence of positive linear operators acting on the space of locally integrable functions. In particular, we obtain rates of ordinary convergence of the sequence of these operators.

Keywords: A-density; A-statistical convergence; Sequence of positive linear operators; Locally integrable function; Modulus of continuity; The Korovkin approximation theorem

1. Introduction

In this work, we are concerned with rates of A-statistical convergence of a sequence of positive linear operators defined on the space of locally integrable functions. The first section of the work introduces some basic ideas related to statistical convergence and the space of locally integrable functions while the second section describes the rates of A-statistical convergence of a sequence of positive linear operators acting on the space of locally integrable functions. The final section addresses the ordinary rates of convergence.

Let A be a nonnegative regular summability matrix [1], and let K be a subset of $\mathbb{N}$, the set of all natural numbers. The A-density of K is defined by $\delta_A(K) := \lim_j \sum_{n=1}^{\infty} a_{jn}\chi_K(n)$ provided the limit exists, where χ_K is the characteristic function of K. Then the sequence $x := (x_n)$ is said to be A-statistically convergent to the number L if, for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$; or equivalently $\lim_j \sum_{n:|x_n-L|\geq\varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim x = L$ [2–4]. If we take $A = C_1$, the Cesáro matrix, then C_1-statistical convergence reduces to statistical convergence [5,6].

Now let $\mathbb{R}$ denote the set of real numbers. Throughout the work we will use the weight function q defined by $q(x) = 1 + x^2$ ($x \in \mathbb{R}$). Then, by $L_{p,q}(\text{loc})$, we denote the space of all locally integrable functions, that is the space of all measurable functions f for which $(\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|^p \mathrm{d}t)^{\frac{1}{p}} \leq M_f q(x)$, $x \in \mathbb{R}$, where M_f is a positive constant

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depending on f and $p \geq 1$. It is known [7] that $L_{p,q}(\text{loc})$ is a linear normed space with the norm

$$\|f\|_{p,q} := \frac{\sup_{x\in\mathbb{R}} \left(\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}}{q(x)},$$

where $\|f\|_{p,q}$ may also depend on $h > 0$. For any real numbers $a, b, (a < b)$, we write $\|f; L_p(a,b)\| = (\frac{1}{b-a}\int_a^b |f(t)|^p \,\mathrm{d}t)^{\frac{1}{p}}$ and $\|f; L_{p,q}(a,b)\| = \sup_{a\leq x\leq b} \frac{\|f;L_p(x-h,x+h)\|}{q(x)}$. With this notation the norm in $L_{p,q}(\text{loc})$ may be written in the form $\|f\|_{p,q} = \sup_{x\in\mathbb{R}} \frac{\|f;L_p(x-h,x+h)\|}{q(x)}$. As usual, if T is a positive linear operator from $L_{p,q}(\text{loc})$ into $L_{p,q}(\text{loc})$, then the operator norm $\|T\|$ is given by $\|T\| := \sup_{f\neq 0} \|Tf\|_{p,q} / \|f\|_{p,q}$.

In [8], using the functional analytic technique the authors have proved some Korovkin type approximation theorems via A-statistical convergence (see also [9–11]).

The main goal of the present work is to study rates of A-statistical convergence of the sequence of operators studied in Lemma 1 and Theorem 3 of [8] by means of the modulus of continuity. To achieve this we use the concepts of rates of A-statistical convergence introduced in [9] (see also [11]). Note that the classical Korovkin type approximation theory may be found in [12,13].

2. Rates of A-statistical convergence

In the classical summability setting, rates of summation have been introduced in several ways (see, e.g., [14,15]). The concept of statistical rates of convergence, for two nonvanishing null sequences, is studied in [16]. Unfortunately no single definition seems to have become the "standard" for the comparison of rates of summability transforms. For this reason various ways of defining rates of convergence in the A-statistical sense were first introduced in [9]. We should recall that those definitions may also be found in [11].

Also, we consider the following weighted modulus of continuity:

$$w_q(f,\delta) = \sup_{|x-y|\leq\delta} \frac{|f(y)-f(x)|}{q(x)},$$

where δ is a positive constant and $f \in L_{p,q}(\text{loc})$. It is easy to see that, for any $c > 0$ and all $f \in L_{p,q}(\text{loc})$,

$$w_q(f, c\delta) \leq (1+[c])w_q(f,\delta), \tag{1}$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

To obtain our main results we first need the following lemma.

Lemma 2.1. *Let $\{T_n\}$ be a sequence of positive linear operators acting from $L_{p,q}(\text{loc})$ into $L_{p,q}(\text{loc})$. Then for each $n \in \mathbb{N}$ and $\delta > 0$, and for every function f that is continuous and bounded on the whole real axis, we have*

$$\begin{aligned}\|T_n f - f; L_{p,q}(a,b)\| &\leq C_1 w_q(f,\delta)\,\|T_n f_0 - f_0\|_{p,q} + C_1 w_q(f,\delta)\\ &\quad + \frac{C_1}{\delta^2} w_q(f,\delta)\,\|T_n \varphi_x\|_{p,q} + C_2\|T_n f_0 - f_0\|_{p,q},\end{aligned}$$

where $f_0(t) := 1$, $\varphi_x(t) := (t-x)^2$, $C_1 := \sup_{a\leq x\leq b} q(x)$, and $C_2 := \sup_{a\leq x\leq b} |f(x)|$.

Proof. Let f be any continuous and bounded function on the whole real axis, and let $x \in [a,b]$ be fixed. Using linearity and monotonicity of T_n, for all $n \in \mathbb{N}$ and for any $\delta > 0$, by (1), we get

$$\begin{aligned}|T_n(f,x) - f(x)| &\leq q(x) T_n\left(w_q\left(f, \frac{|t-x|}{\delta}\delta\right), x\right) + |f(x)|\,|T_n(f_0,x) - f_0(x)|\\ &\leq q(x) w_q(f,\delta)\,|T_n(f_0,x) - f_0(x)| + q(x) w_q(f,\delta)\\ &\quad + \frac{q(x)w_q(f,\delta)}{\delta^2} T_n(\varphi_x, x) + |f(x)|\,|T_n(f_0,x) - f_0(x)|.\end{aligned}$$

Now let $C_1 := \sup_{a \le x \le b} q(x)$ and $C_2 := \sup_{a \le x \le b} |f(x)|$. Then we have

$$\left\|T_n f - f; L_{p,q}(a,b)\right\| \le C_1 w_q(f,\delta)\, \|T_n f_0 - f_0\|_{p,q} + C_1 w_q(f,\delta) + \frac{C_1}{\delta^2} w_q(f,\delta)\, \|T_n \varphi_x\|_{p,q} + C_2 \|T_n f_0 - f_0\|_{p,q},$$

which yields the result. ■

Theorem 2.2. *Let $A = (a_{jn})$ be a nonnegative regular summability matrix, and let (a_n) and (b_n) be positive nonincreasing sequences. Let $\{T_n\}$ be a sequence of positive linear operators from $L_{p,q}(\mathrm{loc})$ into $L_{p,q}(\mathrm{loc})$. Assume that, for each continuous and bounded function f on the real line, the following conditions hold:*

(i) *$\|T_n f_0 - f_0\|_{p,q} = st_A - o(a_n)$, as $n \to \infty$;*
(ii) *$w_q(f, \alpha_n) = st_A - o(b_n)$, as $n \to \infty$ with $\alpha_n = \sqrt{\|T_n \varphi_x\|_{p,q}}$.*

Then we have $\left\|T_n f - f; L_{p,q}(a,b)\right\| = st_A - o(c_n)$, as $n \to \infty$, where $c_n := \max\{a_n, b_n\}$. Similar results hold when little "o" is replaced by big "O".

Proof. Choosing $\delta = \alpha_n = \sqrt{\|T_n \varphi_x\|_{p,q}}$ in Lemma 2.1 we immediately get, for every $n \in \mathbb{N}$, that

$$\left\|T_n f - f; L_{p,q}(a,b)\right\| \le C_1 w_q(f,\alpha_n)\, \|T_n f_0 - f_0\|_{p,q} + 2C_1 w_q(f,\alpha_n) + C_2 \|T_n f_0 - f_0\|_{p,q}. \tag{2}$$

Given $\varepsilon > 0$ define the following sets: $D = \left\{n \in \mathbb{N} : \left\|T_n f - f; L_{p,q}(a,b)\right\| \ge \varepsilon\right\}$,

$$D_1 = \left\{n \in \mathbb{N} : w_q(f,\alpha_n)\, \|T_n f_0 - f_0\|_{p,q} \ge \frac{\varepsilon}{3C_1}\right\},$$

$$D_2 = \left\{n \in \mathbb{N} : w_q(f,\alpha_n) \ge \frac{\varepsilon}{6C_1}\right\}, \qquad D_3 = \left\{n \in \mathbb{N} : \|T_n f_0 - f_0\|_{p,q} \ge \frac{\varepsilon}{3C_2}\right\}.$$

It follows from (2) that $D \subseteq D_1 \cup D_2 \cup D_3$. Also define the sets

$$D_1' = \left\{n \in \mathbb{N} : w_q(f,\alpha_n) \ge \sqrt{\frac{\varepsilon}{3C_1}}\right\}, \qquad D_1'' = \left\{n \in \mathbb{N} : \|T_n f_0 - f_0\|_{p,q} \ge \sqrt{\frac{\varepsilon}{3C_1}}\right\}.$$

Then observe that $D_1 \subseteq D_1' \cup D_1''$. So we have $D \subseteq D_1' \cup D_1'' \cup D_2 \cup D_3$. Now, since $c_j = \max\{a_j, b_j\}$, we get, for every $j \in \mathbb{N}$, that

$$\frac{1}{c_j} \sum_{n \in D} a_{jn} \le \frac{1}{b_j} \sum_{n \in D_1'} a_{jn} + \frac{1}{a_j} \sum_{n \in D_1''} a_{jn} + \frac{1}{b_j} \sum_{n \in D_2} a_{jn} + \frac{1}{a_j} \sum_{n \in D_3} a_{jn}. \tag{3}$$

Letting $j \to \infty$ in (3), and using (i) and (ii), we have $\lim_j \frac{1}{c_j} \sum_{n \in D} a_{jn} = 0$. This means that $\|T_n f - f; L_{p,q}(a,b)\| = st_A - o(c_n)$, as $n \to \infty$, whence the result. ■

The above proof can easily be modified to prove the following analog.

Theorem 2.3. *Let $A = (a_{jn}), (a_n), (b_n), (\alpha_n)$ and $\{T_n\}$ be the same as in Theorem 2.2. Assume that, for each continuous and bounded function f on the real line, the conditions $\|T_n f_0 - f_0\|_{p,q} = st_A - o_\mu(a_n)$ and $w_q(f,\alpha_n) = st_A - o_\mu(b_n)$ (as $n \to \infty$) hold. Then we have $\left\|T_n f - f; L_{p,q}(a,b)\right\| = st_A - o_\mu(d_n)$, as $n \to \infty$, where $d_n := \max\{a_n, b_n, a_n b_n\}$. Similar results hold when little "o_μ" is replaced by big "O_μ".*

We now study the rates of A-statistical convergence in the space of locally integrable functions.

Theorem 2.4. *Let $A = (a_{jn}), (a_n), (b_n), (c_n), (\alpha_n)$ and $\{T_n\}$ be the same as in Theorem 2.2. Assume that the operator norm sequence $\{\|T_n\|\}$ is A-statistically bounded, i.e., $\delta_A(E) = 1$ with $E := \{n \in \mathbb{N} : \|T_n\| \le H\} = 1$ for some $H > 0$. Assume further that, for each function $f \in L_{p,q}(\mathrm{loc})$, the following conditions hold:*

(i) *$\|T_n f_0 - f_0\|_{p,q} = st_A - o(a_n)$, as $n \to \infty$;*
(ii) *$w_q(f,\alpha_n) = st_A - o(b_n)$, as $n \to \infty$.*

Then we get, for any $h > 0$, that $\sup_{x \in \mathbb{R}} \left(\frac{\|T_n f - f; L_p(x-h,x+h)\|}{q^*(x)}\right) = st_A - o(c_n)$, as $n \to \infty$, where q^* is a weight function such that $\lim_{|x| \to \infty} \frac{q(x)}{q^*(x)} = 0$. Similar conclusions are satisfied when little "o" is replaced by big "O".

Proof. By hypothesis, given $\varepsilon > 0$, there exists x_0 such that for all x with $|x| \geq x_0$ we have $\frac{q(x)}{q^*(x)} < \varepsilon$. Let $f \in L_{p,q}(\text{loc})$. Then by Lusin's theorem we can find a continuous function ξ on $[-x_0 - h, x_0 + h]$ such that $\left\| f - \xi; L_p(-x_0 - h, x_0 + h)\right\| < \varepsilon$. Now let $u_n := \sup_{x\in\mathbb{R}}(\frac{\|T_n f - f; L_p(x-h,x+h)\|}{q^*(x)})$. Then it is shown in the proof of Theorem 3 of [8], for every $n \in E$, that

$$u_n \leq K\varepsilon + q(x_0+1)\|T_n G - G;\ L_{p,q}(-x_0, x_0)\|, \tag{4}$$

where $K = (H+1)q(x_0+1) + H$, and the function G is given by

$$G(x) := \begin{cases} \xi(-x_0 - h), & \text{if } x \leq -x_0 - h \\ \xi(x), & \text{if } |x| \leq x_0 + h \\ \xi(x_0 + h), & \text{if } x \geq x_0 + h. \end{cases}$$

Observe that G is continuous and bounded on the whole real axis. Hence, replacing f by G and taking $\delta = \alpha_n = \sqrt{\|T_n \varphi_x\|_{p,q}}$ in Lemma 2.1, it follows from (4), for every $n \in E$, that

$$\begin{aligned} u_n \leq\ & K\varepsilon + C_1' q(x_0+1) w_q(f,\alpha_n)\, \|T_n f_0 - f_0\|_{p,q} \\ & + 2C_1' q(x_0+1) w_q(f,\alpha_n) + C_2' q(x_0+1)\|T_n f_0 - f_0\|_{p,q}, \end{aligned} \tag{5}$$

where $C_1' := \sup_{-x_0\leq x\leq x_0} q(x)$ and $C_2' := \sup_{-x_0\leq x\leq x_0} |f(x)|$. Now given $r > 0$, choose $\varepsilon > 0$ such that $K\varepsilon < r$. Then, as in the proof of Theorem 2.2, define the following sets: $U = \{n \in E : u_n \geq \varepsilon\}$,

$$\begin{aligned} U_1 &= \left\{ n \in E : w_q(f,\alpha_n)\, \|T_n f_0 - f_0\|_{p,q} \geq \frac{r - K\varepsilon}{3C_1' q(x_0+1) w_q(f,\alpha_n)} \right\}, \\ U_2 &= \left\{ n \in E : w_q(f,\alpha_n) \geq \frac{r - K\varepsilon}{6C_1' q(x_0+1)} \right\}, \\ U_3 &= \left\{ n \in E : \|T_n f_0 - f_0\|_{p,q} \geq \frac{r - K\varepsilon}{3C_2' q(x_0+1)} \right\}. \end{aligned}$$

By (5) we have $U \subseteq U_1 \cup U_2 \cup U_3$. Also defining

$$\begin{aligned} U_1' &= \left\{ n \in E : w_q(f,\alpha_n) \geq \sqrt{\frac{r - K\varepsilon}{3C_1' q(x_0+1) w_q(f,\alpha_n)}} \right\}, \\ U_1'' &= \left\{ n \in E : \|T_n f_0 - f_0\|_{p,q} \geq \sqrt{\frac{r - K\varepsilon}{3C_1' q(x_0+1) w_q(f,\alpha_n)}} \right\} \end{aligned}$$

it is clear that $U_1 \subseteq U_1' \cup U_1''$. So we get $U \subseteq U_1' \cup U_1'' \cup U_2 \cup U_3$. Since $c_j = \max\{a_j, b_j\}$, we obtain, for every $j \in \mathbb{N}$, that

$$\frac{1}{c_j}\sum_{n\in U} a_{jn} \leq \frac{1}{b_j}\sum_{n\in U_1'} a_{jn} + \frac{1}{a_j}\sum_{n\in U_1''} a_{jn} + \frac{1}{b_j}\sum_{n\in U_2} a_{jn} + \frac{1}{a_j}\sum_{n\in U_3} a_{jn}. \tag{6}$$

Taking the limit as $j \to \infty$ in (6), and using hypotheses (i) and (ii), we see that $\lim_j \frac{1}{c_j}\sum_{n\in D} a_{jn} = 0$, which completes the proof. ∎

Replacing "o" by "o_μ" one can get the following result immediately.

Theorem 2.5. *Let $A = (a_{jn})$, (a_n), (b_n), (α_n), q^* and $\{T_n\}$ be the same as in Theorem 2.4. Assume that, for each function $f \in L_{p,q}(\text{loc})$, the following conditions: $\|T_n f_0 - f_0\|_{p,q} = st_A - o_\mu(a_n)$ and $w_q(f,\alpha_n) = st_A - o_\mu(b_n)$ (as $n \to \infty$) hold. Then we get, for any $h > 0$, that $\sup_{x\in\mathbb{R}}(\frac{\|T_n f - f; L_p(x-h,x+h)\|}{q^*(x)}) = st_A - o(d_n)$, as $n \to \infty$, where $d_n := \max\{a_n, b_n, a_n b_n\}$. Furthermore, similar conclusions hold when little "o_μ" is replaced by big "O_μ".*

3. Concluding remarks

We first note that if we choose $a_n = 1$ and $b_n = 1$ for each $n \in \mathbb{N}$, then Lemma 1 in [8] may be deduced from Theorem 2.2 or from Theorem 2.3. Similarly, Theorem 3 in [8] may also be deduced from Theorem 2.4 or from Theorem 2.5. So our theorems in Section 2 give us the rate of A-statistical convergence of the sequence of operators studied in [8]. Furthermore, if we replace the matrix $A = (a_{jn})$ by the identity matrix, then Theorem 2.2, Theorem 2.3 and Theorem 2.4, Theorem 2.5 immediately give the following ordinary rates of convergence, respectively.

Corollary 3.1. *Let* $\{T_n\}$ *be a sequence of positive linear operators from* $L_{p,q}(\text{loc})$ *into* $L_{p,q}(\text{loc})$. *If, for each continuous and bounded function* f *on the real line, the conditions* $\lim_n \|T_n f_0 - f_0\|_{p,q} = 0$ *and* $\lim_n w_q(f, \alpha_n) = 0$ *hold, then we have* $\lim_n \|T_n f - f; L_{p,q}(a, b)\| = 0$.

Corollary 3.2. *Let* $\{T_n\}$ *be a sequence of positive linear operators from* $L_{p,q}(\text{loc})$ *into* $L_{p,q}(\text{loc})$ *such that* $\{\|T_n\|\}$ *is bounded. If, for each function* $f \in L_{p,q}(\text{loc})$, *the conditions* $\lim_n \|T_n f_0 - f_0\|_{p,q} = 0$ *and* $\lim_n w_q(f, \alpha_n) = 0$ *hold, then we get, for any* $h > 0$, *that* $\lim_n\{\sup_{x\in\mathbb{R}}(\frac{\|T_n f - f; L_p(x-h,x+h)\|}{q^*(x)})\} = 0$ *where* q^* *is the weight function defined as in Theorem 2.4.*

We note that, in [8], an example of a sequence of positive linear operators has been provided so that our results hold but the classical ones do not.

In the end, we should remark that our primary focus in this work was on dealing with positive linear operators. However, it is still an open question how useful the A-statistical summability concept is in convergence of a sequence of nonlinear operators.

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