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Szász–Mirakjan type operators providing a better error estimation

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Abstract

In this work, giving a modification of the well-known Szasz–Mirakjan operators, we prove that the error estimation of our operators is better than that of the classical Szasz–Mirakjan operators. Furthermore, we obtain a Voronovskaya type theorem for these modified operators.

Keywords: Szász–Mirakjan operators; The Korovkin type approximation theorem; Modulus of continuity; The Voronovskaya type theorem

1. Introduction

Most of the approximating operators, L_n, preserve $e_i(x) = x^i (i = 0, 1)$, i.e., $L_n(e_0; x) = e_0(x)$ and $L_n(e_1; x) = e_1(x)$, $n \in \mathbb{N}$. These conditions hold, specifically, for the Bernstein polynomials, the Szász–Mirakjan operators, and the Baskakov operators (see [1–5]). For each of these operators, $L_n(e_2; x) \neq e_2(x) = x^2$. Recently, King [6] presented a non-trivial sequence $\{V_n\}$ of positive linear operators which approximate each continuous function on [0, 1] while preserving the functions e_0 and e_2. Then it is proved that the operators V_n have a better rate of convergence than the classical Bernstein polynomials whenever $0 \leq x < 1/3$. Statistical variants of King's results have recently been studied by Duman and Orhan [7]. Recently, some approximation results on the Meyer–König and Zeller type operators preserving x^2 have been investigated by the authors [8].

The aim of this work is to obtain a sequence of positive linear operators which has a better rate of convergence than the classical Szász–Mirakjan operators.

We first consider the Banach lattice

$$E := \left\{ f \in C[0, +\infty) : \lim_{x\to+\infty} \frac{f(x)}{1+x^2} \text{ is finite} \right\}$$

endowed with the norm

$$\|f\|_* := \sup_{x\in[0,+\infty)} \frac{|f(x)|}{1+x^2}.$$

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Then, the set $\{e_0, e_1, e_2\}$ is a K_+-subset of E; also the space E is isomorphic to $C[0, 1]$ (see, for details, [2]).

Let us now recall that the well-known Szász–Mirakjan operators are defined on the space E as follows:

$$S_n(f; x) = \mathrm{e}^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}, \quad f \in E \text{ and } x \geq 0. \tag{1}$$

Note that the series on the right hand side of (1) is absolutely convergent because $f \in E$. Furthermore, every S_n maps $C_B[0, +\infty)$, the space of all bounded and continuous functions on $[0, +\infty)$, into itself.

Now, for the Szász–Mirakjan operators, the following lemma is known (see, for instance, [1,9]).

Lemma A. *Let $e_i(x) = x^i$, $i = 0, 1, 2, 3, 4$. Then, for each $x \geq 0$, we have*

(a) $S_n(e_0; x) = 1$,
(b) $S_n(e_1; x) = x$,
(c) $S_n(e_2; x) = x^2 + \frac{x}{n}$,
(d) $S_n(e_3; x) = x^3 + \frac{3x^2}{n} + \frac{x}{n^2}$,
(e) $S_n(e_4; x) = x^4 + \frac{6x^3}{n} + \frac{7x^2}{n^2} + \frac{x}{n^3}$.

2. Construction of the operators

Let $\{u_n(x)\}$ be a sequence of real-valued continuous functions defined on $[0, \infty)$ with $0 \leq u_n(x) < \infty$. Now consider the following operators:

$$D_n(f; x) = \mathrm{e}^{-nu_n(x)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nu_n(x))^k}{k!}, \tag{2}$$

where $x \in [0, \infty)$, $f \in E$ and $n \in \mathbb{N}$. It is clear that the operators D_n are positive and linear. Observe that choosing $u_n(x) = x$, our operators D_n turn out to be the classical Szász–Mirakjan operators given by (1).

Now, if we replace $u_n(x)$ by $u_n^*(x)$ defined as

$$u_n^*(x) := \frac{-1 + \sqrt{4n^2x^2 + 1}}{2n}, \quad x \geq 0 \text{ and } n \in \mathbb{N}, \tag{3}$$

then the operators D_n given by (2) reduce to the operators

$$D_n^*(f; x) = \mathrm{e}^{-nu_n^*(x)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{\left(nu_n^*(x)\right)^k}{k!}, \quad f \in E \text{ and } x \geq 0. \tag{4}$$

Then, observe that every D_n^* maps $C_B[0, +\infty)$ into itself. Hence, it follows from Lemma A that

Lemma 2.1. *For each $x \geq 0$, we have*

(a) $D_n^*(e_0; x) = 1$,
(b) $D_n^*(e_1; x) = -\frac{1}{2n} + \frac{\sqrt{4n^2x^2+1}}{2n}$,
(c) $D_n^*(e_2; x) = x^2$,
(d) $D_n^*(e_3; x) = \frac{3x^2}{2n} + \frac{1}{2n^3} + \left(\frac{x^2}{2n} - \frac{1}{2n^3}\right)\sqrt{4n^2x^2 + 1}$,
(e) $D_n^*(e_4; x) = x^4 + \frac{1}{2n^4} + \left(\frac{2x^2}{n^2} - \frac{1}{2n^4}\right)\sqrt{4n^2x^2 + 1}$.

Now, fix $b > 0$ and consider the lattice homomorphism $T_b : C[0, +\infty) \to C[0, b]$ defined by $T_b(f) := f|_{[0,b]}$ for every $f \in C[0, +\infty)$. In this case, we see that, for each $i = 0, 1, 2$,

$$\lim_{n\to\infty} T_b\left(D_n^*(e_i)\right) = T_b(e_i) \quad \text{uniformly on } [0, b]. \tag{5}$$

On the other hand, with the universal Korovkin type property with respect to monotone operators (see Theorem 4.1.4 (vi) of [2, p. 199]) we have the following: "Let X be a compact set and H be a cofinal subspace of $C(X)$. If E is a Banach lattice, $S : C(X) \to E$ is a lattice homomorphism and if $\{L_n\}$ is a sequence of positive linear operators from $C(X)$ into E such that $\lim_{n\to\infty} L_n(h) = S(h)$ for all $h \in H$, then $\lim_{n\to\infty} L_n(f) = f$ provided that f belongs to the Korovkin closure of H".

Hence, by using (5) and the above property we obtain the following result.

Theorem 2.2. $\lim_{n\to\infty} D_n^*(f) = f$ *uniformly on* $[0, b]$ *provided* $f \in E$ *and* $b > 0$.

In order to get uniform convergence on $[0, +\infty)$ of the sequence $\{D_n^*(f)\}$ we consider the following subspace E^* of E:

$$E^* := \{f \in C[0, +\infty) : \lim_{x\to+\infty} f(x) \text{ is finite}\}$$

endowed with the sup-norm.

For a given $\lambda > 0$, consider the function $f_\lambda(x) := e^{-\lambda x}$, $(x \geq 0)$. Then, for every $x \geq 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} D_n^*(f_\lambda; x) &= \mathrm{e}^{-nu_n^*(x)} \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda \frac{k}{n}} \frac{\left(nu_n^*(x)\right)^k}{k!} \\ &= \mathrm{e}^{-nu_n^*(x)} \sum_{k=0}^{\infty} \frac{\left(nu_n^*(x)\mathrm{e}^{-\lambda/n}\right)^k}{k!} \\ &= \exp\left\{-nu_n^*(x)\left[1 - \exp(-\lambda/n)\right]\right\}. \end{aligned}$$

Observe that

$$\lim_{n\to\infty} D_n^*(f_\lambda) = f_\lambda \quad \text{uniformly on } [0, +\infty).$$

Hence using this limit and applying Proposition 4.2.5-(7) of [2, p. 215] one can obtain the next result at once.

Theorem 2.3. $\lim_{n\to\infty} D_n^*(f) = f$ *uniformly on* $[0, +\infty)$ *provided* $f \in E^*$.

3. Better error estimation

In this section we compute the rate of convergence of the operators D_n^* defined by (4). Then, we will show that our operator has better error estimation than that of the classical Szász–Mirakjan operators S_n given by (1). To achieve this we use the modulus of continuity.

If we define the function ψ_x, $(x \geq 0)$, by $\psi_x(y) = y - x$, then by Lemma 2.1 one can get the following result, immediately.

Lemma 3.1. *For every* $x \geq 0$*, we have*

(a) $D_n^*(\psi_x; x) = -x - \frac{1}{2n} + \frac{\sqrt{4n^2x^2+1}}{2n}$

(b) $D_n^*(\psi_x^2; x) = 2x^2 + \frac{x}{n} - \frac{x\sqrt{4n^2x^2+1}}{n}$,

(c) $D_n^*(\psi_x^3; x) = -4x^3 + \frac{1}{2n^3} + \left(\frac{2x^2}{n} - \frac{1}{2n^3}\right)\sqrt{4n^2x^2 + 1}$,

(d) $D_n^*(\psi_x^4; x) = 8x^4 - \frac{4x^3}{n} - \frac{2x}{n^3} + \frac{1}{2n^4} + \left(-\frac{4x^3}{n} + \frac{2x^2}{n^2} + \frac{2x}{n^3} - \frac{1}{2n^4}\right)\sqrt{4n^2x^2 + 1}$.

Let $f \in C_B[0, +\infty)$ and $x \geq 0$. Then, the modulus of continuity of f denoted by $\omega(f, \delta)$ is defined to be

$$\omega(f, \delta) = \sup_{\substack{|y-x|\leq\delta \\ x,y\in[0,+\infty)}} |f(y) - f(x)|.$$

Then we have the following

Theorem 3.2. *For every $f \in C_B[0,+\infty)$, $x \geq 0$ and $n \in \mathbb{N}$, we have*

$$\left|D_n^*(f;x) - f(x)\right| \leq 2\omega(f,\delta_x),$$

where $\delta_x := \sqrt{2x\left(x - u_n^*(x)\right)}$ and $u_n^*(x)$ is given by (3).

Proof. Now, let $f \in C_B[0,+\infty)$ and $x \geq 0$. Using linearity and monotonicity of D_n^* we easily get, for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$\left|D_n^*(f;x) - f(x)\right| \leq \omega(f,\delta)\left\{1 + \frac{1}{\delta}\sqrt{D_n^*\left(\psi_x^2;x\right)}\right\},$$

Now applying Lemma 3.1(b) and choosing $\delta = \delta_x$ the proof is completed. ■

Remark. For the classical Szász–Mirakjan operators S_n given by (1) we know that, for every $f \in C_B[0,+\infty)$, $x \geq 0$ and $n \in \mathbb{N}$,

$$|S_n(f;x) - f(x)| \leq 2\omega(f,\alpha_x), \tag{6}$$

where $\alpha_x := \sqrt{\frac{x}{n}}$.

Now we claim that the error estimation in Theorem 3.2 is better than that of (6) provided $f \in C_B[0,+\infty)$ and $x \geq 0$. Indeed, for $x \geq 0$ and $n \in \mathbb{N}$, we have $\sqrt{4n^2x^2+1} \geq 2nx$, which yields that

$$-\frac{1}{2n} + \frac{\sqrt{4n^2x^2+1}}{2n} \geq x - \frac{1}{2n}. \tag{7}$$

It follows from (3) and (7) that $x - u_n^*(x) \leq \frac{1}{2n}$. This guarantees that $\delta_x \leq \alpha_x$ for $x \geq 0$, which corrects our claim.

4. A Voronovskaya type theorem

In this section, we prove a Voronovskaya type theorem for the operators D_n^* given by (4).

We first need the following lemma.

Lemma 4.1. $\lim_{n\to\infty} n^2 D_n^*\left(\psi_x^4;x\right) = 3x^2$ *uniformly with respect to $x \in [0,b]$ with $b > 0$.*

Proof. Then, by using (3) in Lemma 3.1(d) and after some simple calculations, we may write that

$$\begin{aligned} n^2 D_n^*\left(\psi_x^4;x\right) &= -\frac{4nx^3}{2nx+\sqrt{4n^2x^2+1}} + \frac{2x^2}{2nx+\sqrt{4n^2x^2+1}} \\ &\quad + 2x\left(\frac{-1+\sqrt{4n^2x^2+1}}{n}\right) + \frac{1-\sqrt{4n^2x^2+1}}{2n^2}. \end{aligned}$$

Now taking limit as $n \to \infty$ on both sides of the above equality we obtain

$$\lim_{n\to\infty} n^2 D_n^*\left(\psi_x^4;x\right) = -x^2 + 0 + 4x^2 + 0 = 3x^2$$

uniformly with respect to $x \in [0,b]$, $(b > 0)$, which completes the proof. ■

Theorem 4.2. *For every $f \in E$ such that $f', f'' \in E$, we have*

$$\lim_{n\to\infty} n\left\{D_n^*(f;x) - f(x)\right\} = \frac{1}{2}xf''(x) - \frac{1}{2}f'(x)$$

uniformly with respect to $x \in [0,b]$, $(b > 0)$.

Proof. Let $f, f', f'' \in E$. Define

$$\Psi(y,x) = \begin{cases} \dfrac{f(y) - f(x) - (y-x)f'(x) - \frac{1}{2}(y-x)^2 f''(x)}{(y-x)^2}, & \text{if } y \neq x \\ 0, & \text{if } y = x. \end{cases}$$

Then by assumption we have $\Psi(x, x) = 0$ and the function $\Psi(\cdot, x)$ belongs to E. Hence, by Taylor's theorem we get

$$f(y) = f(x) + (y - x) f'(x) + \frac{(y - x)^2}{2} f''(x) + (y - x)^2 \Psi(y, x).$$

Now from Lemma 3.1(a)–(b)

$$n\{D_n^*(f; x) - f(x)\} = n \left(x - u_n^*(x)\right) \left(x f''(x) - f'(x)\right) + n D_n^* \left(\psi_x^2(y) \Psi(y, x); x\right). \tag{8}$$

If we apply the Cauchy–Schwarz inequality for the second term on the right hand side of (8), then we conclude that

$$n \left| D_n^* \left(\psi_x^2(y) \Psi(y, x); x\right) \right| \le \left(n^2 D_n^*(\psi_x^4(y); x)\right)^{\frac{1}{2}} \left(D_n^*(\Psi^2(y, x); x)\right)^{\frac{1}{2}}. \tag{9}$$

Let $\eta(y, x) := \Psi^2(y, x)$. In this case, observe that $\eta(x, x) = 0$ and $\eta(\cdot, x) \in E$. Then it follows from Theorem 2.2 that

$$\lim_{n\to\infty} D_n^* \left(\Psi^2(y, x); x\right) = \lim_{n\to\infty} D_n^* \left(\eta(y, x); x\right) = \eta(x, x) = 0 \tag{10}$$

uniformly with respect to $x \in [0, b]$, $(b > 0)$. Now considering (9) and (10), and also using Lemma 4.1, we immediately see that

$$\lim_{n\to\infty} n D_n^* \left(\psi_x^2(y) \Psi(y, x); x\right) = 0 \tag{11}$$

uniformly with respect to $x \in [0, b]$. On the other hand, observe now that, by (3),

$$\lim_{n\to\infty} n \left(x - u_n^*(x)\right) = \frac{1}{2}. \tag{12}$$

Then, taking limit as $n \to \infty$ in (8) and using (11) and (12) we have

$$\lim_{n\to\infty} n \left\{D_n^*(f; x) - f(x)\right\} = \frac{1}{2} \left(x f''(x) - f'(x)\right)$$

uniformly with respect to $x \in [0, b]$. The proof is completed. ■

Finally, as in Theorem 2.3, one can obtain the following result at once.

Theorem 4.3. *For every $f \in E^*$ such that $f', f'' \in E^*$, we have*

$$\lim_{n\to\infty} n \left\{D_n^*(f; x) - f(x)\right\} = \frac{1}{2} x f''(x) - \frac{1}{2} f'(x)$$

uniformly with respect to $x \in [0, +\infty)$.

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