**TOBB UNIVERSITY OF ECONOMICS AND TECHNOLOGY**
**INSTITUTE OF NATURAL AND APPLIED SCIENCES**

# APPROXIMATE RESULTS FOR NON-LINEAR CRAMÉR-LUNDBERG TYPE RISK MODEL

**M.Sc. THESIS**

**Yusup ALLYYEV**

**Department of Industrial Engineering**

**Supervisor: Prof. Dr. Tahir HANALİOĞLU**

**AUGUST 2021**

**ÖZET**

Yüksek Lisans

DOĞRUSAL OLMAYAN CRAMÉR-LUNDBERG TİPİ RİSK MODELİ İÇİN YAKLAŞIK SONUÇLAR

Yusup ALLYYEV

TOBB Ekonomi ve Teknoloji Üniveritesi
Fen Bilimleri Enstitüsü
Endüstri Mühendisliği Anabilim Dalı

Danışman: Prof. Dr. Tahir HANALİOĞLU

Tarih: Ağustos 2021

Bu çalışmada doğrusal olmayan bir Cramér-Lundberg risk modeli ele alınmış, araştırılmış ve iflas olasılıkları, $\psi(u)$, hesaplanmıştır. Literatürde klasik model olarak da bilinen bu modelin doğrusal gösterimi şu şekilde tanımlanır:

$$\boldsymbol{U(t) = u + ct - S(t)} \qquad (1)$$

Denklem (1)'deki $U(t)$ risk süreci, belirli bir $t$ zamanında bir sigorta şirketinin sermaye miktarını ifade eder, sabit $u$ şirketin başlangıç sermayesidir, $c$ – prim oranı, $S(t) = \sum_{i=1}^{N(t)} X_i$, $[0, t]$ aralığında meydana gelen kazalar için yapılan ödemelerden dolayı sermaye çıkışını tanımlayan bir ödüllü-yenileme sürecidir, $N(t)$ bir yenileme süreci olup $[0, t]$ aralığındaki toplam kaza sayısını belirtmektedir, $X_i$'ler ise, $i$. hasar için ödeme miktarını gösteren bağımsız ve aynı dağılıma sahip rasgele değişkenlerdir. Denklem (1)'de görüldüğü gibi şirketin prim gelirini ifade eden $\boldsymbol{ct}$ terimi zamanın doğrusal bir fonksiyonudur. Ancak bu varsayım gerçekçi değildir, çünkü bir sigorta şirketinin prim geliri her zaman doğrusal olarak artamaz. Bu, özellikle sigorta poliçesi sahipleri ile doymuş pazarlar için geçerlidir. Bu nedenle, prim gelirinin, monoton olarak artmasına rağmen, büyüme hızı zamanla azalan bir

fonksiyon olarak modellenmesi tavsiye edilir. Bu nedenle, bu çalışmada aşağıdaki gibi ifade edilen, daha gerçekçi özel bir doğrusal olmayan matematiksel model inşa edilmiş ve incelenmiştir:

$$\boldsymbol{V(t) = u + c\sum_{i=1}^{N(t)} ln(1 + W_i) + c\,ln\big(1 + (t - T_{N(t)})\big) - S(t)} \qquad (2)$$

Denklem (2)'de, $W_i$'lar $(i = 1, 2, 3 \dots)$ kazalar arasındaki süreleri gösteren pozitif, bağımsız ve aynı dağılıma sahip rastgele değişkenler dizisidir; $T_{N(t)} = \sum_{i=1}^{N(t)} W_i$ ise, $W_i$ , $i = 1,2,3, \dots$ rastgele değişkenlerinin dizisine karşılık gelen bir ödüllü-yenileme sürecidir ve *Logaritmik Risk Süreci* olarak adlandırılan $V(t)$ ise herhangi bir $t$ zamanda şirketin sermaye dengesini tanımlar. Bu çalışmanın temel amacı, denklem (2)'deki doğrusal olmayan risk modelinin iflas etme olasılığını, $\psi(u)$, hesaplamaktır. Model oluşturulurken stokastik süreçler, yenileme süreçleri, ödüllü-yenileme süreçleri ve bu süreçlerin olasılıksal özellikleri kullanılmıştır. İlk aşamada doğrusal olmayan modelimizin iflas etme olasılığı için Lundberg tipi üst sınır bulunmuştur. Bu olasılık sınırları hesaplanmaya çalışılırken doğrusal olmayan denklemlerle karşılaşıldığında sayısal çözüm yöntemleri kullanılmıştır. Çeşitli senaryoları dikkate almak için farklı olasılık dağılımları ve parametreleri göz önünde bulundurup, regresyon modeli ile yaklaşık bir çözüm bulunmuştur. İkinci aşamada, bu doğrusal olmayan model için yukarıdan ve aşağıdan iflas olasılığının yaklaşık sınırları bulunmuştur. Bu aynı zamanda iflas olasılığı için Cramér tipi sınır olarak da bilinir. Bu amaçla, kazaları (hasarları) temsil eden $\{X_n\}$ dizisi tarafından üretilen yenileme sürecinin kalan ömrünün limit dağılımını tanımlayan rastgele değişkeni $\hat{X}$'ın istatistiksel özelliklerinden yararlanılmıştır. Özellikle, iflas olasılığının sınır ifadesinde bilinmeyen bir katsayı olan sabit bir $C$'yi belirlemek için $\hat{X}$'ın moment çıkaran fonksiyonu kullanılmıştır. Bu ifadeleri sadeleştirmek ve kompakt bir forma dönüştürmek için kalkülüs yöntemleri kullanılmıştır. Benzer şekilde, iflas olasılıklarını incelemek ve hesaplamak için farklı senaryoları dikkate almak için çeşitli olasılık dağılımları ve parametreler kullanılmıştır.

**Anahtar Kelimeler:** Risk Teorisi, Cramér-Lundberg Risk Modeli, Doğrusal Olmayan Sigorta Modeli, İflas Olasılığı, Lundberg Eşitsizliği.

## ABSTRACT

Master of Science

APPROXIMATE RESULTS FOR NON-LINEAR CRAMÉR-LUNDBERG TYPE RISK MODEL

Yusup ALLYYEV

TOBB University of Economics and Technology
Institute of Natural and Applied Sciences
Industrial Engineering Science Programme

Supervisor: Prof. Dr. Tahir HANALİOĞLU

Date: August 2021

In this study, a non-linear Cramér-Lundberg risk model is considered, investigated and ruin probabilities, $\psi(u)$, are calculated. In literature, a linear form of this model, also known as classical model, is defined as follows:

$$\boldsymbol{U(t) = u + ct - S(t)} \tag{1}$$

The risk process $U(t)$ in Eq.(1) expresses an amount of capital of an insurance company at a given time $t$, the constant $u$ is initial capital of the company, $c$ – the premium rate, $S(t) = \sum_{i=1}^{N(t)} X_i$ is a renewal-reward process which represents the outflow of cash caused by reimbursements for claims occurred in the interval $[0, t]$, $N(t)$ is a renewal process counting the total number of claims within the time frame $[0, t]$ and $X_i$'s are i.i.d. random variables denoting the amount of payment for $i^{th}$ claim. As seen in (1), the term $\boldsymbol{ct}$ expressing the company's premium income is a linear function of time. However, this assumption is not realistic, because the premium income of an insurance company cannot always increase linearly. This is especially true for the markets saturated with insurance policy holders. Therefore, it is advisable to assume that the premium income is modeled as a function whose rate of growth decreases with time, although this function is monotonically increasing. For this reason, in this work, a more realistic special non-linear mathematical model is constructed and investigated, which is given as follows:

$$\boldsymbol{V(t) = u + c\sum_{i=1}^{N(t)} ln(1 + W_i) + c\, ln\big(1 + (t - T_{N(t)})\big) - S(t)} \tag{2}$$

In (2), $W_i$'s $(i = 1, 2, 3 \dots)$ are positive i.i.d. sequence of random variables describing inter-arrival times of claims; $T_{N(t)} = \sum_{i=1}^{N(t)} W_i$ is a renewal-reward process, corresponding to the sequence of random variables $W_i's$ , $i = 1,2,3, \dots$, and $V(t)$ defines company's capital balance at any time $t$ which is modelled by so called a *Logarithmic Risk Process*. The main purpose of this study is to evaluate ruin probability, $\psi(u)$, of non-linear risk model in (2). While establishing the model, stochastic processes, renewal processes, reward-renewal processes and the probabilistic characteristics of these processes were used. In the first stage, the Lundberg type upper bound was found for the ruin probability of our non-linear model. While trying to calculate these probability bounds, numerical solution methods were used when nonlinear equations were encountered. In order to consider various scenarios, different probability distributions and parameters are considered and an approximate solution is found with the regression model. In the second stage, bounds for ruin probability from above and below is found for this non-linear model. This is also known as Cramér-type bound for the ruin probability. For this purpose, the statistical characteristics of the random variable, $\hat{X}$, which describes the residual time (limit distribution) of the renewal process produced by the sequence $\{X_n\}$, representing the accidents(damages), was exploited. In particular, moment generating function of $\hat{X}$ was utilized to determine a constant $C$, which is an unknown coefficient in the bound expression of the ruin probability. In order to simplify these expressions and transform them into a compact form, calculus methods were used. Similarly, in order to examine and calculate ruin probabilities, various probability distributions and parameters were used to consider different scenarios.

**Keywords:** Risk Theory, Cramér-Lundberg Risk Model, Non-Linear Insurance Model, Ruin Probability, Lundberg's Inequality.

**CONTENTS**

## LIST OF FIGURES

**Page**

## LIST OF TABLES

**Page**

## ABBREVIATIONS

| | |
|---|---|
| **c.d.f.** | **:** cumulative distribution function |
| **i.i.d.** | **:** independent, identically distributed |
| **NPC** | **:** Net Profit Condition |
| **p.d.f.** | **:** probability density function |

## LIST OF SYMBOLS

Symbols used in this work are presented below with their explanations.

| **Symbols** | **Explanation** |
|---|---|
| $E_F X$ | expectation of $X$ with respect to the distribution $F$ |
| $Exp(\lambda)$ | Exponential distribution with parameter $\lambda$: $F(x) = 1 - e^{-\lambda x}$ , $x > 0$ |
| $F$ | distribution function/distribution of a random variable |
| $F_A$ | distribution function/distribution of the random variable $A$ |
| $F_I$ | integrated tail distribution: $F_I(x) = (E_F X)^{-1} \int_0^x \bar{F}\,(y) dy,\ \ x \geq 0$ |
| $\bar{F}$ | tail of the distribution function $F$: $\bar{F} = 1 - F$ |
| $F^{n*}$ | $n$-fold convolution of the distribution function/distribution $F$ |
| $N, N(t)$ | claim number or claim number process |
| $(\Omega, \mathcal{F}, \mathrm{P})$ | probability space |
| $\psi(u)$ | ruin probability |
| $\rho$ | safety loading |
| $S_n$ | cumulative sum of $X_1, X_2, \ldots\ X_n$ |
| $S, S(t)$ | total, aggregate claim amount process |
| $t$ | time, index of a stochastic process |
| $T_i$ | arrival times of a claim number process |
| $u$ | initial capital |
| $U[a, b]$ | uniform distribution on $(a, b)$ |
| $U(t)$ | risk process |
| $V(t)$ | non-linear risk process |
| $var(X)$ | variance of the random variable $X$ |
| $X_n$ | claim size |
| $\sim$ | $X \sim F$: $X$ has distribution $F$ |
| $\approx$ | approximately equals |

# 1. INTRODUCTION

There are many applications of stochastic processes in science and engineering. *Risk Theory* is one of them. In actuarial science and applied probability, another term for risk theory is *non-life insurance mathematics*, which models insurance business. Theft, fires, floods, accidents, riots etc. – are natural or man-made examples of catastrophes that can be investigated in the framework of stochastic processes. In particular, it models the claims that occur and recommends to insurance businesses how much premium to charge to policy holders so that the insurance company does not run into insolvency/ruin. One of the prominent models in risk theory is Cramér-Lundberg Risk model.

## 1.1 The Fundamental Model

At the onset of the nineteenth ($19^{th}$) century, Filip Lundberg (1903), the Swedish actuary, introduced a simple model. In this model, the basic relationship between incoming cash premiums and outgoing claim amounts is established. In literature, this model is known as Cramér-Lundberg Risk model. The equation describing this *risk process* (also known as surplus process) is as follows:

$$\boldsymbol{U(t) = u + p(t) - S(t), \quad t \geq 0} \tag{1.1}$$

where the terms are defined as follows:

$\boldsymbol{U(t)}$ **:** Insurance company's cash balance at time $t$ – risk process

$\boldsymbol{u = U(0) > 0}$ **:** Company's initial capital at $t = 0$ – constant

$\boldsymbol{p(t) \equiv ct}$ **:** Premium income function. $\boldsymbol{c} > 0$ is the premium rate

$\boldsymbol{S(t) \equiv \sum_{i=1}^{N(t)} X_i}$**:** Reward renewal process describing the outflowing reimbursements due to claims happened in the time frame $[0, t]$ – total claim amount process

$\boldsymbol{X_i}$ **:** positive i.i.d. random variables denoting the amount of payment for the $i^{th}$ claim, for $i = 1,2,3$ ...

$\boldsymbol{N(t)}$ **:** Renewal process describing the number of claims occurred in $[0, t]$ – claim number process, the counting process

$$N(t) \equiv max\left\{n \geq 1 : \ T_n \ = \sum_{i=1}^{n} W_i \ \leq t \ , t \geq 0\right\}$$

$\boldsymbol{T_i}$ **:** Claim arrival times

$$T_0 = 0 \, , T_n = \ W_1 + \cdots + W_n \ \text{ and } 0 \leq T_1 \leq T_2 \dots .$$

$\boldsymbol{W_i}$ **:** Positive i.i.d. random variables denoting the inter-arrival times of claims, for $i = 1{,}2{,}3 \dots$

The i.i.d. property of the claim sizes, $X_i$, presumes that the portfolio at hand has a homogenous structure. This means that the portfolio of insurance policies (contracts) carries similar risks such as insurance against fire of suburban houses or against accident of particular type of cars.

Figure 1.1 below helps to better visualize the risk process, by providing a sample path of both processes $N$ and $S$, which are corresponding counting process and compound sum process, respectively. Observe that both process' jumps occur at times when $t = T_i$, for $i = 1, 2, 3, \dots$ at discrete times. By an amount of 1 in the former case and by an amount of $X_i$ for the former case. Also, (1.1) can be written in more explicit and expressive form as in (1.2) below:

$$\boldsymbol{U(t) = u + c \cdot t \ - \sum_{i=1}^{N(t)} X_i} \tag{1.2}$$

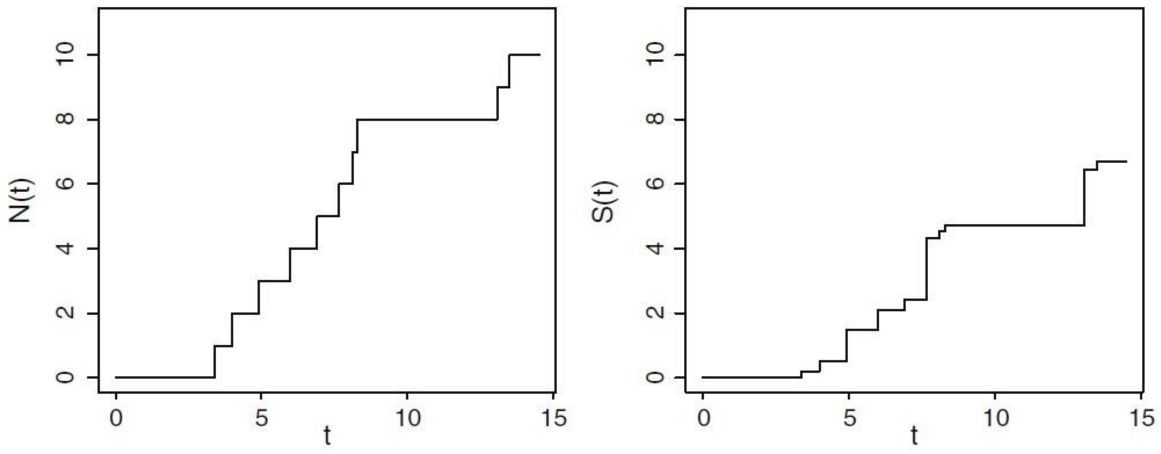


Figure 1.1 : A corresponding sample path of the process $N$ and $S$

The graph in Figure 1.2 below, depicts the idealized evolution of the process $U(t)$. With initial capital $u$, the process $U(t)$ increases linearly in each time interval $[T_i, T_{i+1})$, for $i = 0, 1, 2 \dots$ , with slope $c$, until the disruption time when the accident occurs. To be specific, let us take time interval $[T_0, T_1)$, in which the process has grown linearly wih the slope $c$ up until an accident disrupts the process by an amount of $X_1$, and hence the value of $U(t)$ has decreased by that exact amount. Similarly, the process regains the upward movement with the slope $c$ exactly at time $T_1$ up until the process is interrupted by the second accident at $T_2$ with an intensity of $X_2$, so on and so forth.

It is also possible that the process $U(t)$ could assume a negative values, in the case of when accident occurs with a large claim size, sufficiently large so that it can cause the process $U(t)$ to fall below zero. In that situation, we call this event as ***ruin***.

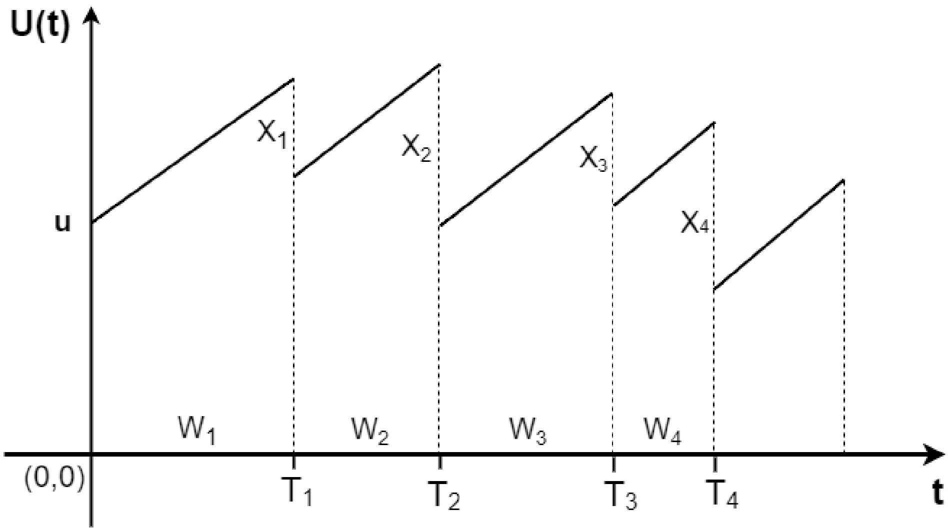


Figure 1.2: An idealized realization of the risk process $U(t)$

## 1.2 Definitions and Theoretical Background

In this sub-section we elaborate on the total claim amount process $S(t)$

$$\boldsymbol{S(t)} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{N(t)}} \boldsymbol{X_i}\ , \quad \boldsymbol{t \geq 0}, \tag{1.3}$$

in which we assume that $N(t)$ and $(X_i)$ are independent, in other words, the claim number process and the claim size sequence are independent. Also, note that $(X_i)$ is independent and identically distributed (i.i.d.) and positive, i.e., $X_i > 0$ , sequence of random variables. By the choice of the type of the the process $N(t)$, various realizations for the process $S(t)$ can be modelled.

### 1.2.1 The Reason Behind the Linearity of Income Function $\boldsymbol{p(t)}$

In order to decide how much premium to charge, one needs to determine the shape of the premium income function $p(t)$. This should be done in accordance with the order of magnitude of the process $S(t)$, which models the aggregate losses of the risk process $U(t)$. The determination of the shape of the $p(t)$ is important, bacause on one hand, if one chooses $p(t)$ to be too steep, the insurance company may not be competitive due to overly premium charges. On the other hand, if one chooses $p(t)$ to be too slowly increasing function, the insurance company runs a risk of ruin.

In general, it is analytically complicated to determine the distribution of $S(t)$, in other words, it can be mathematically intractble problem. Therefore, researchers utilize simulation methods or numerical calculations with the aim of estimation of the distribution of $S(t)$. But in this section, we aim for the basic idea about the size f the total claim amount. Hence we look at the elementary properties of $S(t)$, such as its expectation and its variance. These characteristics of $S(t)$ can be determined by the well-known techniques which are the strong law of large numbers and the central limit theorem, respectively, for $S(t)$ when time $t$ tends to infinity.

We assumed that $(X_i)$ and $N(t)$ are independent. Also assume that $E\big(N(t)\big) < \infty$ and $E(X_1) < \infty$. Then, the expecation of $S(t)$ can be calculated as follows:

$E\big(S(t)\big) = E\Big[E(\sum_{i=1}^{N(t)} X_i \,\Big|N(t))\Big] = E(N(t)E(X_1)) = E(N(t))E(X_1)$

We know from renewal theory that, in the basic model, $EN(t) = \lambda t$, when $N(t)$ is a homogenous Poisson process with intensity of $\lambda$ parameter. Hence,

$$E\big(S(t)\big) = \lambda t E(X_1)$$

This result tells us only about the average behaviour of the process $S(t)$, i.e., when time tends to infinity, the expected total claim grows roughly linearly with respect to time in the simple model.

In order to gain more insight about the behaviour of $S(t)$ furthar than its expected value, one could consider its vairance. Let us assume that $var\big(N(t)\big) < \infty$ and $var(X_1) < \infty$. Then, by using the property of independence of $N(t)$ and $(X_i)$, we can derive the following two equations:

i. $var\Big[\sum_{i=1}^{N(t)} X_i \,\Big|N(t)\Big] = \sum_{i=1}^{N(t)} var\big(X_i\big|N(t)\big) = N(t)var\big(X_1\big|N(t)\big)$

$= N(t)var(X_1)$

ii) $E\Big[\sum_{i=1}^{N(t)} X_i \,\Big|N(t)\Big] = N(t)E(X_1)$

Note that, conditional $N(t)$ only implies the choice of type of $N(t)$ in our derivation.

Thus, the variance of $S(t)$ can be written as follows:

$$\begin{aligned} var\big(S(t)\big) &= E[N(t)var(X_1)] + var\big(N(t)E(X_1)\big) \\ &= E\big(N(t)\big)var(X_1) + var(N(t))(E(X_1))^2. \end{aligned}$$

**Proposition 1**

In the renewal model, if $E(W_1) = \lambda^{-1} < \infty$ and $E(X_1) < \infty$ ,

$$\lim_{t \to \infty} \frac{E(S(t))}{t} = \lambda E(X_1) \ ,$$

and if $\text{var}(W_1) < \infty$ and $\text{var}(X_1) < \infty$,

$$\lim_{t \to \infty} \frac{\text{var}(S(t))}{t} = \lambda[\text{var}(X_1) + \text{var}(W_1)\lambda^2(E(X_1))^2] \ .$$

For proof of the proposition, please refer to Mikosch (2004), Lemma 2.3.4.

In summary, we have the following:

$$E\big(S(t)\big) = \lambda\, t\, E(X_1) \ \text{ and } \ \text{var}\big(\text{S(t)}\big) = \lambda\, t\, E({X_1}^2)$$

for every $t > 0$ , in the Cramér-Lundberg model.

In a nutshell, this result implies that in the renewal model, the expectation and the variance of the total claim amount grow roughly linearly as a function of $t$. This information sheds light on how to charge premiums. As can be seen from results, the premium should increase roughly linearly and with a slope larger than $\lambda E X_1$ in order to compensate for accidents represented by $S(t)$. And this summarizes the reason behind conventional assumption that premium income function $p(t)$ is linear in time.

### 1.2.2 Classical Types of Calculation of Premium Principles

In order for insurance companies to avoid ruin, a proper premium charging policy is necessary. The collected premiums will be used to cover losses, represented by $S(t)$, that occured over time.

The simplest approach would be considering the expectation of the process $S(t)$. Recall that,

$$E\big(S(t)\big) = \lambda\, t\, E(X_1)$$

and

$$p(t) = ct$$

Then, by inspection it can be said that the company loses when $p(t) < E(S(t))$ and gains profit when $p(t) > E(S(t))$, on average, for large $t$.

Therefore, it is reasonable to *"load"* the expected total claim amount by $\rho$, which is a certain positive number. As an example, consider the renewal model in Proposition 1, in which we derived the following equation:

$$E\big(S(t)\big) = \lambda E(X_1)t\big(1 + o(1)\big),\ t \to \infty\ .$$

Then one can choose $p(t)$ accordingly as below:

$$p(t) = (1+\ \rho)E\big(S(t)\big) \quad \text{or} \quad p(t) = (1+\ \rho)\lambda E(X_1)t$$

for some positive number $\rho$, which is known as the *safety loading* in the literature**.**

The choice of $\rho$ is done considering the balance between the possibility of the ruin of the insurance company versus the risk of being less competitive in the market. In other words, if one chooses $\rho$ large, the premiums would be higher and hence the insurance business will be making relatively more money and hence avoiding the ruin. On the other hand, if $\rho$ is too large, the insurance company runs the risk of losing customers (the insureds), as the other insurance companies might be offering lower premiums, which eventually results in the loss of the premium income. In short, the choice of the premium charge is an intricate topic. Preferably, the company should find an optimal balance between premium income and the total claim amount. Therefore, more sophisticated principles must be developed when it comes to premium calculation. Only some of the well-known principles in the literature are discussed below:

**i) The net or equivalence principle**: $p_{net}(t)$

This is a benchmark premium as the name suggests. Because, the premium $p(t)$ is determined as follows:

$$p_{net}(t) = E(S(t))\ .$$

i.e., the premium is exactly equal to the expectation of the total claim amount $S(t)$.

Which means, on average, the company does not gain or lose cash. This can be interpreted as fair market premium, but it would be misleading, becasue this choice of $p(t)$ does not take into consideration the fact that the process $S(t)$ has non-zero variance and its deviation from its mean can be both in the positive and negative directions. Therefore, the company should avoid this principle, since this could cause some losses to the company and eventually force to run into ruin. Therefore, not a prudent choice from company's perspective.

**ii) The expected value principle:** $p_{EV}(t)$

$$p_{EV}(t) = (1 + \rho)E(S(t)) ,$$

for some $\rho > 0$. The reasoning behind this principle is similar to that of in (i).

**iii) The variance principle**

$$p_{var}(t) = E\big(S(t)\big) + \alpha \operatorname{var}\big(S(t)\big),$$

for some $\alpha > 0$.

The rationale behind this principle is in an asymptotic sense that Propositipon 1 suggests. Here, $\alpha$ serves as positive safety loading as in the case (ii) above.

**iv) The standard deviation principle:** $p_{SD}(t)$

$$p_{SD}(t) = E\big(S(t)\big) = \alpha \sqrt{\operatorname{var}(S(t))} ,$$

for some $\alpha > 0$.

Observe that, by choice of the principle, one charges a smaller premium when compared to the expected value and variance principles.

## 1.3 Unrealistic Assumption – Need For More Realistic Model

There are mainly two assumptions about this classical model:

Assumption 1: $p(t)$ is deterministic and linear.

Assumption 2: The claim size process $(X_i)$ and the claim arrival process $(T_i)$ are ***mutually independent***.

Although assumption 1 might hold for some rare cases, in general it is not realistic in real life. Because the premium income of an insurance company cannot always increase linearly. Therefore, it is advisable to assume that the premium income is modeled as a function whose rate of growth decreases with time, although this function is monotonically increasing. For this reason, in this thesis, a more realistic special non-linear mathematical model is constructed and investigated.

## 2. LITERATURE REVIEW

Insurance and insurance business is a concept that concerns and affects almost everyone in our daily lives. Therefore, insurance is an inevitable part of developed economies. Contemporary economies and modern states would hardly operate without insurance companies. Because, these institutions guarantee compensation to almost any actors of the society at the individualistic, company, or the organizational level at an unfortunate times when catastrophes such as fires, floods, accidents and riots befalls onto them. The notion of insurance is interesting topic. It is an integral part of our civilization. The trust of the insurer and the insured runs the businesses. Therefore, the examination of risk and ruin problems of insurance company has an important role in actuarial science. Many valuable studies have been done in the literature on this subject.

It is crucial that science must be at the core of this mutual trust between the insurer and the insured. Therefore, in the $20^{th}$ century, Filip Lundberg [25] and Harald Cramér [10], the Swedish mathematicians and the pioneers in this area, laid foundations of modern risk theory based on the probability theory, statistics, and stochastic processes. They realized that the theory of stochastic processes provides the most appropriate framework for modeling an insurance business, i.e., modelling the claims and their inter-arrival times. In recent years, the Cramér-Lundberg model is one of the pillars of non-life insurance mathematics [28]. This model has been extended and adapted to various fields of applies probability: queuing theory, renewal theory, branching processes, reliability, financial mathematics and extreme value theory are just some of them.

The literature on risk theory is vast. There are many models that aim at describing the risk process. They all have one thing in common which is extending or modifying the classical model, which is given in (2.1) below:

$$\boldsymbol{U(t) = u + p(t) - S(t), \quad t \geq 0} \qquad (2.1)$$

where the terms are defined as follows:

$\boldsymbol{U(t)}$ **:** Insurance company's cash balance at time $t$ – risk process

$\boldsymbol{u = U(0) > 0}$ **:** Company's initial capital at $t = 0$ – constant

$\boldsymbol{p(t) \equiv ct}$ **:** Premium income function where positive $c$ is the premium rate

$\boldsymbol{S(t)}$**:** Reward renewal process describing the outgoing reimbursements due to accidents happened in the time frame $[0, t]$ – total claim amount process

In classical model, there are mainly three components comprising the governing equation of risk process $U(t)$: initial capital $u$, premium income $p(t)$, and total claim amount process $S(t)$, as can be seen in (2.1). Almost each related work on the topic contains assumption(s) about one or more components of the equation and some kind of relationship between or among them. These assumptions are merely for the purpose of capturing the real-life scenario of insurance business, while still being able to track the problem mathematically and computationally.

There are few assumptions, other than when interest rates are assumed to be non-zero or non-constant, about initial capital $u$ – as it is a constant amount of initial reserve [37]. The question of how much of initial capital is required to keep the probability of ruin above some predefined threshold value is answered in [27]. In a very few cases investment of the initial capital $u$ as income or preventive measure is considered [29]. The evaluation of ruin probabilities strongly depends on the distribution of the claim amounts, and given two or more claim distributions, it is natural to ask which one implies larger values of ruin probabilities in finite or infinite time horizon for the given initial capital value of $u$. This issue is addressed in [8, 23, 26, 32, 36]. As far as the author is concerned, the majority of the research is done on variations of premium income function $p(t)$ and its relation to the total claim process $S(t)$. A study conducted in [4] deals with stochastic premium income function which is also independent of the risk process. A scenario when the total claim amount process is the same as in the classical model while the premium income, unlike the classical case, is considered to be a stochastic process, called as random premiums risk process, is investigated in [35] and ruin probabilities are estimated numerically. Similarly, in [38], risk model with stochastic premiums income is considered and some specific dependence structure among the claim sizes, inter-claim times and premium sizes is assumed.

The studies mentioned above are all valuable, each focusing on specific sets of conditions. However, to the best of authors knowledge, there is no work that models premium income function $p(t)$ as a general non-linear (deterministic) function except that of in [19]. In which, the premium income function was assumed to be root-square function, i.e., $p(t) = c\sqrt{t}$. Although, the assumption that $p(t)$ is linear function might hold for some rare cases, in general it is not realistic in real life. Because the premium income of an insurance company cannot always increase linearly. This is especially true for the markets saturated with insurance policy holders. Therefore, it is reasonable to assume that the premium income is modeled as a function whose rate of growth decreases with time, although this function is monotonically increasing.

The contributions of this study are many fold. Firstly, a brand-new non-linear risk process model, $V(t)$, was introduced. In this model, the premium income function could assume any form so long as its rate of growth decreases with time, although it is monotonically increasing. In this study, we assumed $p(t)$ to be a logarithmic function, i.e., $g(t) = cln(1+t)$. And we called this a *Logarithmic Risk Process*, $V(t)$, for obvious reasons. Second, we calculated a Lundberg-type upper bound for ruin probability of this non-linear risk process, utilizing numerical methods. Lastly, we have derived a new exact formula for Cramér-type bound for ruin probability of this risk process, which is both from below and above. While doing so, the statistical characteristics of the random variable, $\hat{X}$, which describes the residual time (limit distribution) of the renewal process produced by the sequence $\{X_n\}$, denoting the accidents(damages), was exploited. In particular, moment generating function of $\hat{X}$ was utilized to determine a constant $C$, which is an unknown coefficient in the bound expression of the ruin probability. And an approximate value of this constant is calculated in conjunction with the previous results. In order to simplify these expressions and transform them into a compact form, calculus methods were used

The rest of the study is organized as follows. In the next section, non-linear model is constructed and some definitions are provided. Following that, two sections are dedicated to calculation of ruin probabilities. And in the last section, conclusion of this study is presented.

# 3. NON-LINEAR MODEL: GENERAL FUNCTIONS

In this section, a special case of non-linear Cramér-Lundberg risk model is considered and investigated. In the previous sections, a linear form of this model was defined as follows:

$$\boldsymbol{U(t) = u + p(t) - S(t)} \tag{3.1}$$

where the terms are defined as follows:

$\boldsymbol{U(t)}$ **:** Insurance company's cash balance at time $t$ – risk process
$\boldsymbol{u = U(0) > 0}$ **:** Company's initial capital at $t = 0$ – constant
$\boldsymbol{p(t) \equiv ct}$ **:** Premium income function. $\boldsymbol{c} > 0$ is the premium rate
$\boldsymbol{S(t)}$**:** Reward renewal process describing the outflowing reimbursements due to claims happened in the time frame $[0, t]$ – total claim amount process

In equation (3.1), $p(t) -$ premium income function, is linear in time, i.e., $p(t) = ct$. However, this assumption is not realistic, because the premium income of an insurance company cannot always increase linearly. This is especially true for the markets saturated with insurance policy holders. Therefore, it is advisable to assume that the premium income is modeled as a function whose rate of growth decreases with time, although this function is monotonically increasing. For this reason, in this work, a more realistic special non-linear mathematical model is constructed and investigated. Although, any general function possessing aforementioned properties is suitable, in this work logarithmic function was preferred to due to its nice analytical properties. A related work was done by Hanalioğlu [19], where premium income function was root square function. Therefore, premium income function for our non-linear model becomes as follows:

$$\boldsymbol{g(t) = cln(1 + t)}$$

instead of $p(t) = ct$. Observe that $g(t)$ is monotonically increasing function whose rate of growth is decreasing with time.

Now we can define *Logarithmic Risk Process*, $V(t)$, as follows:

$$\boldsymbol{V(t) = u + c \sum_{i=1}^{N(t)} ln(1 + W_i) + c\, ln\big(1 + (t - T_{N(t)})\big) - S(t)} \tag{3.2}$$

where the terms are defined as follows:

$\boldsymbol{V(t)}$ **:** Insurance company's cash balance at time $t$ – non-linear risk process

$\boldsymbol{u = U(0) > 0}$ **:** Company's initial capital at $t = 0$ – constant

$\boldsymbol{g(t) \equiv c ln(1 + t)}$ **:** Premium income function. $\boldsymbol{c} > 0$ is the premium rate

$\boldsymbol{S(t) \equiv \sum_{i=1}^{N(t)} X_i}$**:** Reward renewal process describing the outflowing reimbursements due to claims happened in the time frame $[0, t]$ – total claim amount process

$\boldsymbol{X_i}$ **:** positive i.i.d. random variables denoting the amount of payment for the $i^{th}$ claim, for $i =$ 1,2,3 ...

$\boldsymbol{N(t)}$ **:** Renewal process describing the number of claims occurred in $[0, t]$ – claim number process, the counting process

$$N(t) \equiv max\left\{n \geq 1 : \ T_n = \sum_{i=1}^{n} W_i \ \leq t \ , t \geq 0\right\}$$

$\boldsymbol{T_i}$ **:** Claim arrival times

$$T_0 = 0 \,, T_n = W_1 + \cdots + W_n \text{ and } 0 \leq T_1 \leq T_2 \dots .$$

$\boldsymbol{W_i}$ **:** positive i.i.d. random variables denoting the inter-arrival times of claims, for $i = 1,2,3$ ...

To better visualize the evolution of the logarithmic risk process, please refer to Figure 3.1 below:

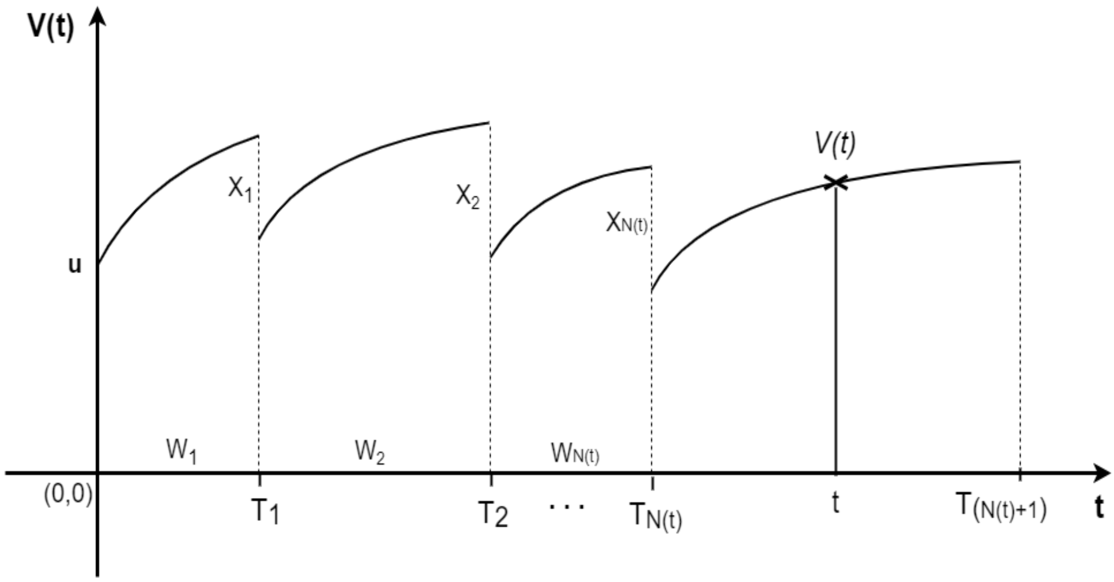


Figure 3.1 : Logarithmic Risk Process $V(t)$

Observe that, now our model has become non-linear, as the name suggests. A visual comparison of Figure 3.1 and Figure 3.2 captures the differences of two processes, where former is non-linear and the latter is linear.

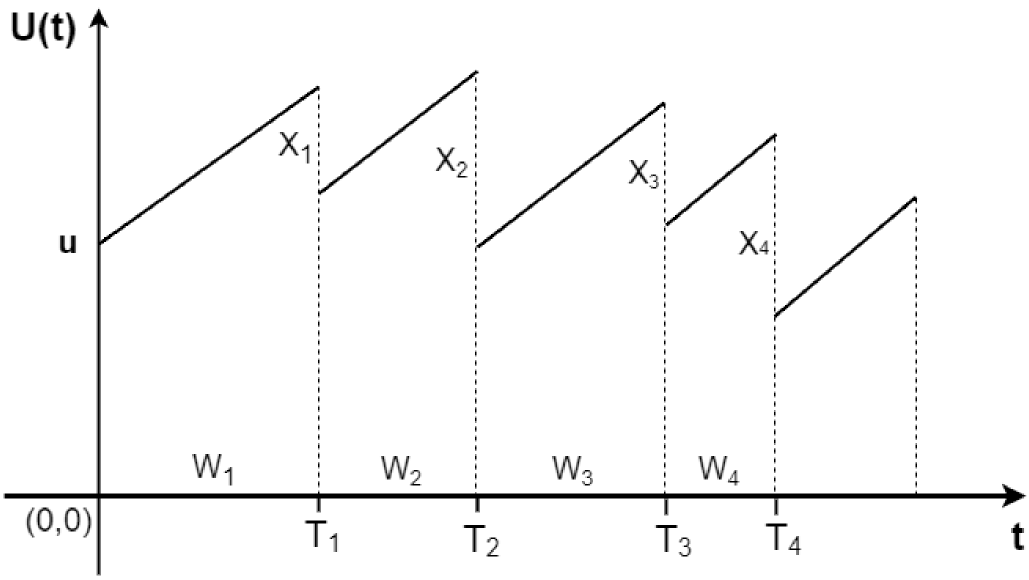


Figure 3.2 : A linear risk process $U(t)$

The main purpose of this thesis is to calculate the ruin probability of the non-linear model in the equation (3.2). For this purpose, we need to establish some related definitions. In the following, three definitions are presented.

**Definition 3.1** (Ruin, ruin time, ruin probability)

i. The event that $V$ ever falls below zero is called *ruin*:

$$\boldsymbol{Ruin} = \{V(t) < 0 \ \ for\ some\ t \ > 0\}$$

ii. The time when the process falls below zero for the first time is called *ruin time*, *T*:

$$T = \inf\{t > 0 : V(t) < 0\}$$

iii. Then $\psi(u)$ – the probability of ruin, is then given by:

$$\boldsymbol{\psi(u)} \equiv P(Ruin \mid V(0) = u) = P(T < \infty), \qquad u > 0$$

Observe that ruin can occur only at discrete times $t = T_n$ for some $n \geq 1$. Therefore, we can write

$$\begin{aligned} Ruin &= \left\{\inf_{t>0} V(t) < 0\right\} = \left\{\inf_{n\geq 1} \ V(T_n) \ < 0\right\} \\ &= \left\{\inf_{n\geq 1} \ [u + g(T_n) - S(T_n)] \ < 0\right\} \\ &= \left\{\inf_{n\geq 1} \ [u + c \textstyle\sum_{i=1}^{n} g(W_i) - \sum_{i=1}^{n} X_i \ ] \ < 0\right\}. \end{aligned}$$

In the last step of the definition of the event ruin we used the fact that

$$N(T_n) = max\{i \geq 1 : \ T_i \leq T_n\} = n \ \ a.s.$$

since we assumed that $W_j > 0$ and $g(W_i) > 0$ a.s. for all $j \geq 1$.

Since, $T_n = W_1 + \cdots + W_n$ , write

$$Z_n = X_n - cg(W_n) , \ \ S_n = Z_1 + \cdots + Z_n , n \geq 1, \quad S_0 = 0 .$$

Then we have the following alternative expression for the ruin probability $\psi(u)$ with initial capital of $u$:

$$\psi(u) = P\left(\inf_{n\geq 1}\ (-S_n)\ <\ -u\right) = P\left(\sup_{n\geq 1}\ S_n > u\right) \tag{3.3}$$

Equation (3.3) can be summarized as: $\psi(u)$ is the tail probability of the supremum functional of the random walk $(S_n)$, because sequences $(W_i)$ and $(X_i)$ are mutually independent and each of the sequences seperately are comprised of i.i.d. random variables.

**Definition 3.2** (Net Profit Condition – NPC )

We say that the renewal model satisfies the net profit condition $(NPC)$ if

$$E(Z_1) = E(X_1) - cE(g(W_1))\ < 0 \tag{3.4}$$

The NPC condition can be interpreted as: the expected claim size $E(X_1)$ has to be smaller than the expected premium income $cE(g(W_1))$, in a given unit of time. In other words, more premium income should be earned by the company than the paid claim sizes by the company. However, this does not mean that ruin of the company is completely averted. Because, net profit conditin does not take into consideration the fluctuational bevaviour of the stochastic process.

In this model, a small claim condition is assumed, meaning that, there exists a moment generating function of the claim size distribution in a neighborhood of the origin, i.e.,

$$m_{X_1}(h) = E(e^{hX_1}), h\ \in (-h_0, h_0)\ \ for\ some\ h_0\ > 0.$$

Also, by Markov's Inequality, we can rewrite the above eqaution as: for $h\ \in (0, h_0)$,

$P(X_1 > x) = P(e^{hX_1} >\ e^{hx}) \leq e^{-hx} m_{X_1}(h), for\ some\ h\ \in (0, h_0)$ and for all $x > 0$

**Definition 3.3** (Generalized Lundberg coefficient)

Assume that there exists a moment generating function of $Z_1$ in some neighborhood of $(-h_0, h_0)$, $for\ h_0 > 0$, of the origin. If there exists a unique positive solution $r > 0$ to the equation below,

$$r \equiv \inf\{h > 0 : \ m_{Z_1}(h) = E(e^{hZ_1}) = E\left(e^{h(X_1 - cg(W_1))}\right) = 1\} \tag{3.5}$$

it is called the *generalized Lundberg coefficient*.

# 4. APPROXIMATION TO LUNDBERG TYPE BOUND FOR NON-LINEAR MODEL

In this part of the thesis, a Lundberg type upper bound is found for our non-linear model constructed previously. The following theorem establishes this relation.

**Theorem 4.1** (Lundberg's inequality)

Assume the non-linear model in (3.2) whch satisfies the net profit condition. And also assume that the generalized adjustment coefficient $r$ exists. Then, for all $u > 0$, the following inequality holds:

$$\boldsymbol{\psi(u) \leq e^{-r \cdot u}}$$

For proof of the Theorem 4.1, please refer to Mikosch (2004)

Observe that the probability of ruin will be very small for a large initial capital $u$, with the exponential bound from above. The magnitude of the adjustment coefficient $r$ is also vital. The smaller r is, the riskier is the portfolio. Also, the result of the theorem is much more informative than the average behavior of the portfolio, as it was also commented in the definition of the NPC. It is assumed that the initial capital is known. In order to find a Lundberg-type upper bound for our model, what remains to be calculated is the generalized Lundberg coefficient $r$, which we do in the following sections.

## 4.1 Investigation and Analysis of the Generalized Lundberg Coefficient $\boldsymbol{r}$

In order to find $r$, let us reconsider the equation (3.5),

$$m_{Z_1}(h) = E(e^{hZ_1}) = E\left(e^{h(X_1 - cg(W_1))}\right) = 1 \tag{4.1}$$

Before we move on, let's remind that $X_i -$ positive i.i.d random variable denoting the amount of payment for the $i^{th}$ claim for $i = 1,2,3 \ldots$ and $g(t) = \ln(1+t)$ – premium income function.

Expanding (4.1) yields the following:

$$m_{Z_1}(h) = E(\exp(hZ_1))|_{h=r} = E\left(\exp\left(r \cdot \left(X_1 - c \cdot g(W_1)\right)\right)\right)$$
$$= E(\exp(r \cdot X_1)) \cdot E\big(\exp\big(-cr \cdot g(W_1)\big)\big)$$

Now equating the right-hand side to 1, we get:

$$\Longrightarrow \underbrace{E(\exp(r \cdot X_1))}_{M_X(r)} \cdot \underbrace{E\big(\exp\big(-cr \cdot g(W_1)\big)\big)}_{M_{g(W)}(-cr)} = 1$$

By definition of the moment generating function (MGF), we can write the following:

$$M_X(r) \cdot M_{g(W)}(-cr) = 1 \tag{4.2}$$

Here, note that $M_X(r)$ is the MGF of a general random variable $X_1$ and $M_{g(W)}$ is the MGF of $g(W_1)$. Since it very hard to compute $r$ for a general random variable, we investigate particular cases. In other words, we resort to well-studied distributions.

**<u>Special Case-1</u>**: $X_1 \sim Exp(\mu)$ and $W_1 \sim Exp(\lambda)$

Let $X_1 \sim Exp(\mu)$. PDF of the exponential random variable $X_1$ is given as follows:

$$f_{X_1}(x) = \begin{cases} \mu e^{-\mu x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Also, MGF of $X_1$ is as follows:

$$M_X(r) = E(\exp(r \cdot X_1)) = \frac{\mu}{\mu - r}, \text{ where } r < \mu$$

Then from equation (4.2), we get :

$$M_X(r) \cdot M_{g(W)}(-cr) = 1 \quad \Longrightarrow \quad M_{g(W)}(-cr) = \frac{\mu - r}{\mu} = 1 - \frac{r}{\mu}$$

For simplicity, define, $f_1(r) \equiv M_{g(W)}(-cr) = E(\exp(-cr \cdot g(W_1)))$ and

$$f_2(r) \equiv \frac{\mu - r}{\mu} = 1 - \frac{r}{\mu}$$

Obviously, we have $f_1(r) = f_2(r)$

From the definition of the moment generating function, we can write the following:

$$f_1(r) \equiv M_{g(W)}(-cr) = \int_0^\infty \exp\big(-cr \cdot g(t)\big)\, dF_W(t) = 1 - \frac{r}{\mu} \equiv f_2(r) \qquad (4.3)$$

in which, $F_W(t)$ is the cumulative distribution function (CDF) of a random variable $W_1$.

Now, assume that $W_1 \sim Exp(\lambda)$ . Then,

$$f_1(r) = \int_0^\infty \lambda \cdot \exp(-\lambda t) \exp\big(-cr \cdot g(t)\big)\, dt =$$

$$= \lambda \int_0^\infty \exp\big(-\lambda t - cr \cdot g(t)\big)\, dt = 1 - \frac{r}{\mu} = f_2(r) \qquad (4.4)$$

Substituting $g(t) = \ln(1 + t)$ into (4.4), one gets:

$$f_1(r) = \lambda \int_0^\infty \exp(-\lambda t - cr \cdot \ln(1 + t))\, dt =$$

$$= \lambda \int_0^\infty \exp(-\lambda t) \cdot \frac{dt}{(1 + t)^{cr}}$$

Since $f_1(r) = f_2(r)$, we can write the following equality:

$$\lambda \int_0^\infty \exp(-\lambda t) \cdot \frac{dt}{(1 + t)^{cr}} = 1 - \frac{r}{\mu} \qquad (4.5)$$

Since our aim is to compute $r$, now question reduces to finding $r$ satisfying in (4.5). Since it hard to find a general solution, we solve for $r$ with some parameter values of $\lambda$ *and* $c$. To better illustrate, let $\lambda = 1$ . Then (4.5) becomes,

$$f_1(r) = \int_0^\infty \exp(-t) \cdot \frac{dt}{(1 + t)^{cr}} = 1 - \frac{r}{\mu} = f_2(r)$$

Now also assume that $c = 0.7$, then (4.5) becomes as follows:

$$f_1(r) = \int_0^{\infty} \frac{e^{-t}}{(1+t)^{0.7\cdot r}} dt = 1 - \frac{r}{\mu} = f_2(r), \quad for\ r > 0 \tag{4.6}$$

Clearly, there is no analytical solution to $r$ in equation (4.6) due to the integral at the left-hand side. Therefore, it should be calculated numerically, which we do in the following section with the help of software package MATLAB. Also, we vary the values of $\mu$ to see how the solution values of $r$ changes respectively.

**Special Case-2**: $X_1 \sim U[0, b]$ and $W_1 \sim Exp(\lambda)$

Let $X_1 \sim U[0, b]$. PDF of the uniform random variable $X_1$ is given as follows:

$$f_{X_1}(x) = \begin{cases} \frac{1}{b}, & 0 \le x \le b \\ 0, & x > b \end{cases}$$

Also, MGF of $X_1$ is as follows:

$$M_X(r) = E(\exp(r \cdot X_1)) = \begin{cases} \frac{e^{br} - 1}{br}, & r \neq 0 \\ 1, & r = 0 \end{cases}$$

Also, assume that $W_1 \sim Exp(\lambda)$ . Then, similar analysis yields:

$$f_1(r) = \int_0^{\infty} \exp(-t) \cdot \frac{dt}{(1+t)^{cr}} = \frac{br}{e^{br} - 1} = f_2(r) \tag{4.7}$$

Again, there is no analytical solution for *r* in this equation. Therefore, it should be calculated numerically in MATLAB.

## 4.2 Computation of $r$

Since there is no analytical closed form solution for $r$ in (4.6) and (4.7), an algorithm was devised on MATLAB which gives numeric solutions to $r$. Roughly, this algorithm finds the intersection of $f_1(r)$ and $f_2(r)$, as we vary the values of $\mu$.

Algorithm steps are given below:

**1.** Solve LHS and RHS for $0 \leq r \leq K,\ for\ some\ 0 < K < \infty$. Then we plot the graphs of both $f_1(r)$ and $f_2(r)$. (Fix $\mu > 0$ value.)

**2.** By visual inspection, we roughly decide the approximate value of $r$ at large increments of $r$.

**3.** Then around that approximate value of $r$, we search for better approximation of $r$ by making the increment sizes of the algorithm smaller.

**4.** Then around that better approximated neighborhood, we start to take differences of $f_1(r)$ and $f_2(r)$. Whenever the difference changes its sign, we decide that point to be a solution to $r$.

**5.** Decrease/increase the value of $\mu$ and repeat the steps (1-4).

In the following section, numeric values of $r$ are calculated for some special cases.

### 4.3 Numerical calculation of $r$ for some special cases

In this section, we tackle the problem numerically with the help of the computer software.

#### 4.3.1 Special Case 1: Exponential claim arrivals and exponential claim size distributions

In section 4.1, we have already established the following equation for Special Case 1 when $X_1 \sim Exp(\mu)$ and $W_1 \sim Exp(\lambda)$, where $\lambda = 1$ and $c = 0.7$

$$f_1(r) = \int_0^{\infty} \frac{e^{-t}}{(1+t)^{0.7r}} \mathrm{dt} = 1 - \frac{r}{\mu} = f_2(r) \tag{4.8}$$

To calculate $r$, the algorithm devised in Section 4.2 was deployed in MATLAB. By varying the value of $\mu$, the corresponding values of $r$ were calculated. In the Tables from 4.1 through 4.7, sample calculations of $r$ is demonstrated.

In the first row of the tables, the related parameters are provided. And in the last row, the computed approximate value of $r$ is provided and the corresponding ruin probability is given.

In the first column of the tables, the number of steps of the algorithm is presented. In the second columns, the value of $r$ is varied and the values of $f_1(r)$ and $f_2(r)$ are presented in the third and the fourth columns of the tables. As the rule of the algorithm, whenever the sign of the difference between $f_1(r)$ and $f_2(r)$ is changed in the fifth columns, there we find the intersection of $f_1(r)$ and $f_2(r)$. In other words, the $r$ value which cause the change in the sign of the difference of $f_1(r)$ and $f_2(r)$ is the solution to the related equation. The two values of $r$ which correspond to the solution is highlighted.

For example, when $\lambda = 1$, $c = 0.7$ and $\mu = 20$, the value of $r = 18.465$. Observe that in order to determine $r$, two values of $r$ where $f_1(r) - f_2(r)$ changes sign is averaged out. Since these are approximate solutions, it is a good practice to take arithmetic average. Because one cannot tell if $r$ should be rounded up or down.

Table 4.1 : The value of $r$ for $\lambda = 1$; $c = 0.7$; $\mu = 20$

| **λ=1 ; c=0.7 ; μ=20 ;** | | | | |
|---|---|---|---|---|
| **n** | **r** | **$f_1$(r)** | **$f_2$(r)** | **$f_1$(r) – $f_2$(r)** |
| **0** | 18.4000 | 0.077143 | 0.080000 | -0.002857 |
| **1** | 18.4100 | 0.077102 | 0.079500 | -0.002398 |
| **2** | 18.4200 | 0.077060 | 0.079000 | -0.001940 |
| **3** | 18.4300 | 0.077019 | 0.078500 | -0.001481 |
| **4** | 18.4400 | 0.076978 | 0.078000 | -0.001022 |
| **5** | 18.4500 | 0.076937 | 0.077500 | -0.000563 |
| **6** | **18.4600** | 0.076896 | 0.077000 | **-0.000104** |
| **7** | **18.4700** | 0.076855 | 0.076500 | **0.000355** |
| **8** | 18.4800 | 0.076813 | 0.076000 | 0.000813 |
| **9** | 18.4900 | 0.076772 | 0.075500 | 0.001272 |
| **10** | 18.5000 | 0.076732 | 0.075000 | 0.001732 |
| $\boldsymbol{r^*} \approx \mathbf{18.465}$ ; ***Ruin Pr***: $\boldsymbol{\psi(u)} \leq \boldsymbol{e^{-18.465 \cdot u}}$ | | | | |

Table 4.2 : The value of $r$ for $\lambda = 1$; $c = 0.7$; $\mu = 15$

| **λ=1 ; c=0.7 ; μ=15 ;** | | | | |
|---|---|---|---|---|
| **n** | **r** | **$f_1$(r)** | **$f_2$(r)** | **$f_1$(r) – $f_2$(r)** |
| **0** | 13.0000 | 0.108465 | 0.133333 | -0.024868 |
| **1** | 13.1000 | 0.107659 | 0.126667 | -0.019008 |
| **2** | 13.2000 | 0.106865 | 0.120000 | -0.013135 |
| **3** | 13.3000 | 0.106082 | 0.113333 | -0.007252 |
| **4** | **13.4000** | 0.105310 | 0.106667 | **-0.001357** |

| | | | | |
|---|---|---|---|---|
| **5** | **13.5000** | 0.104549 | 0.100000 | **0.004549** |
| **6** | 13.6000 | 0.103799 | 0.093333 | 0.010466 |
| **7** | 13.7000 | 0.103060 | 0.086667 | 0.016393 |
| **8** | 13.8000 | 0.102330 | 0.080000 | 0.022330 |
| **9** | 13.9000 | 0.101611 | 0.073333 | 0.028278 |
| **10** | 14.0000 | 0.100902 | 0.066667 | 0.034236 |
| $r^* \approx 13.45$ ; $Ruin\ Pr$: $\psi(u) \leq e^{-13.45 \cdot u}$ | | | | |

Table 4.3 : The value of $r$ for $\lambda = 1; c = 0.7; \mu = 10$

| **λ=1 ; c=0.7 ; μ=10 ;** | | | | |
|---|---|---|---|---|
| **n** | **r** | **$f_1$(r)** | **$f_2$(r)** | **$f_1$(r) – $f_2$(r)** |
| **0** | 8.0000 | 0.172479 | 0.200000 | -0.027521 |
| **1** | 8.1000 | 0.170494 | 0.190000 | -0.019506 |
| **2** | 8.2000 | 0.168551 | 0.180000 | -0.011449 |
| **3** | **8.3000** | 0.166651 | 0.170000 | **-0.003349** |
| **4** | **8.4000** | 0.164792 | 0.160000 | **0.004792** |
| **5** | 8.5000 | 0.162973 | 0.150000 | 0.012973 |
| **6** | 8.6000 | 0.161192 | 0.140000 | 0.021192 |
| **7** | 8.7000 | 0.159449 | 0.130000 | 0.029449 |
| **8** | 8.8000 | 0.157741 | 0.120000 | 0.037741 |
| **9** | 8.9000 | 0.156069 | 0.110000 | 0.046069 |
| **10** | 9.0000 | 0.154430 | 0.100000 | 0.054430 |
| $r^* \approx 8.35$ ; $Ruin\ Pr$: $\psi(u) \leq e^{-8.35 \cdot u}$ | | | | |

Table 4.4 : The value of $r$ for $\lambda = 1; c = 0.7; \mu = 5$

| **λ=1 ; c=0.7 ; μ=5 ;** | | | | |
|---|---|---|---|---|
| **n** | **r** | **$f_1$(r)** | **$f_2$(r)** | **$f_1$(r) – $f_2$(r)** |
| **0** | 2.8000 | 0.4092263 | 0.4400000 | -0.0307737 |
| **1** | 2.9000 | 0.3995585 | 0.4200000 | -0.0204415 |
| **2** | **3.0000** | 0.3902824 | 0.4000000 | **-0.0097176** |
| **3** | **3.1000** | 0.3813770 | 0.3800000 | **0.0013770** |
| **4** | 3.2000 | 0.3728226 | 0.3600000 | 0.0128226 |
| **5** | 3.3000 | 0.3646009 | 0.3400000 | 0.0246009 |
| **6** | 3.4000 | 0.3566948 | 0.3200000 | 0.0366948 |
| **7** | 3.5000 | 0.3490881 | 0.3000000 | 0.0490881 |
| **8** | 3.6000 | 0.3417656 | 0.2800000 | 0.0617656 |

| | | | | |
|---|---|---|---|---|
| **9** | 3.7000 | 0.3347131 | 0.2600000 | 0.0747131 |
| **10** | 3.8000 | 0.3279172 | 0.2400000 | 0.0879172 |
| $\boldsymbol{r^* \approx 3.05}$ ; $\boldsymbol{Ruin\ Pr}$: $\boldsymbol{\psi(u) \leq e^{-3.05 \cdot u}}$ | | | | |

Table 4.5 : The value of $r$ for $\lambda = 1$; $c = 0.7$; $\mu = 3$

| **λ=1 ; c=0.7 ; μ=3 ;** | | | | |
|---|---|---|---|---|
| **n** | **r** | **$f_1$(r)** | **$f_2$(r)** | **$f_1$(r) – $f_2$(r)** |
| **0** | 0.5000 | 0.820122 | 0.833333 | -0.013212 |
| **1** | 0.6000 | 0.790119 | 0.800000 | -0.009881 |
| **2** | **0.7000** | 0.761790 | 0.766667 | **-0.004877** |
| **3** | **0.8000** | 0.735018 | 0.733333 | **0.001684** |
| **4** | 0.9000 | 0.709698 | 0.700000 | 0.009698 |
| **5** | 1.0000 | 0.685732 | 0.666667 | 0.019065 |
| **6** | 1.1000 | 0.663030 | 0.633333 | 0.029696 |
| **7** | 1.2000 | 0.641508 | 0.600000 | 0.041508 |
| **8** | 1.3000 | 0.621091 | 0.566667 | 0.054424 |
| **9** | 1.4000 | 0.601706 | 0.533333 | 0.068372 |
| **10** | 1.5000 | 0.583287 | 0.500000 | 0.083287 |
| $\boldsymbol{r^* \approx 0.75}$ ; $\boldsymbol{Ruin\ Pr}$: $\boldsymbol{\psi(u) \leq e^{-0.75 \cdot u}}$ | | | | |

Table 4.6 : The value of $r$ for $\lambda = 1$; $c = 0.7$; $\mu = 2.4$

| **λ=1 ; c=0.7 ; μ=2.40 ;** | | | | |
|---|---|---|---|---|
| **n** | **r** | **$f_1$(r)** | **$f_2$(r)** | **$f_1$(r) – $f_2$(r)** |
| **0** | 0.0000 | 1.00000000000 | 1.00000000000 | 0.00000000000 |
| **1** | **0.0050** | 0.99791603816 | 0.99791666667 | **-0.00000062850** |
| **2** | **0.0100** | 0.99583856764 | 0.99583333333 | **0.00000523431** |
| **3** | 0.0150 | 0.99376756371 | 0.99375000000 | 0.00001756371 |
| **4** | 0.0200 | 0.99170300173 | 0.99166666667 | 0.00003633506 |
| **5** | 0.0250 | 0.98964485719 | 0.98958333333 | 0.00006152385 |
| **6** | 0.0300 | 0.98759310566 | 0.98750000000 | 0.00009310566 |
| **7** | 0.0350 | 0.98554772285 | 0.98541666667 | 0.00013105619 |
| **8** | 0.0400 | 0.98350868455 | 0.98333333333 | 0.00017535122 |
| **9** | 0.0450 | 0.98147596666 | 0.98125000000 | 0.00022596666 |
| **10** | 0.0500 | 0.97944954519 | 0.97916666667 | 0.00028287852 |
| $\boldsymbol{r^* \approx 0.0075}$ ; $\boldsymbol{Ruin\ Pr}$; $\boldsymbol{\psi(u) \leq e^{-0.0075 \cdot u}}$ | | | | |

Table 4.7 : The value of $r$ for $\lambda = 1; c = 0.7; \mu = 2.375$

| **λ=1 ; c=0.7 ; μ=2.375 ;** | | | | |
|---|---|---|---|---|
| **n** | **r** | **$f_1$(r)** | **$f_2$(r)** | **$f_1$(r) – $f_2$(r)** |
| **0** | 0.0000 | 1.000000000 | 1.000000000 | 0.000000000 |
| **1** | 0.0050 | 0.999582687 | 0.999574468 | 0.000008219 |
| **2** | 0.0100 | 0.999165635 | 0.999148936 | 0.000016699 |
| **3** | 0.0150 | 0.998748843 | 0.998723404 | 0.000025438 |
| **4** | 0.0200 | 0.998332310 | 0.998297872 | 0.000034438 |
| **5** | 0.0250 | 0.997916038 | 0.997872340 | 0.000043698 |
| **6** | 0.0300 | 0.997500026 | 0.997446809 | 0.000053217 |
| **7** | 0.0350 | 0.997084272 | 0.997021277 | 0.000062996 |
| **8** | 0.0400 | 0.996668778 | 0.996595745 | 0.000073034 |
| **9** | 0.0450 | 0.996253544 | 0.996170213 | 0.000083331 |
| **10** | 0.0500 | 0.995838568 | 0.995744681 | 0.000093887 |
| $\boldsymbol{r^* \approx Does\ not\ exist}$ | | | | |

Also notice that in Table 4.7 the solution to $r$ does not exist. This is because the NPC condition is violated. In fact, the solution to $r$ cuts off at around $\mu \approx 2.4$. Let's confirm this numerically as well. We know that NPC should hold, i.e.,

$$E(X_1) - cE(g(W_1)) < 0$$

must hold. We know that $E(X_1) = \frac{1}{\mu}$, since $X_1 \sim Exp(\mu)$. Also, $W_1 \sim Exp(\lambda)$ and $E\big(g(W_1)\big) = \int_0^\infty \ln(1+t) \cdot \lambda e^{-\lambda t} dt = \int_0^\infty \ln(1+t) \cdot e^{-t} dt \approx 0.5963$ was calculated in MATLAB. Since $c = 0.7$, solving for $\mu$, we find that

$$\mu \approx \frac{1}{(0.7) \cdot (0.5963)} = 2.396$$

This is expected due to NPC condition.

Now, let's plot $\mu$ vs $r^*$ in the Figure 5.1. In total 23 different values of $r$ was calculated for 23 different values of $\mu$.

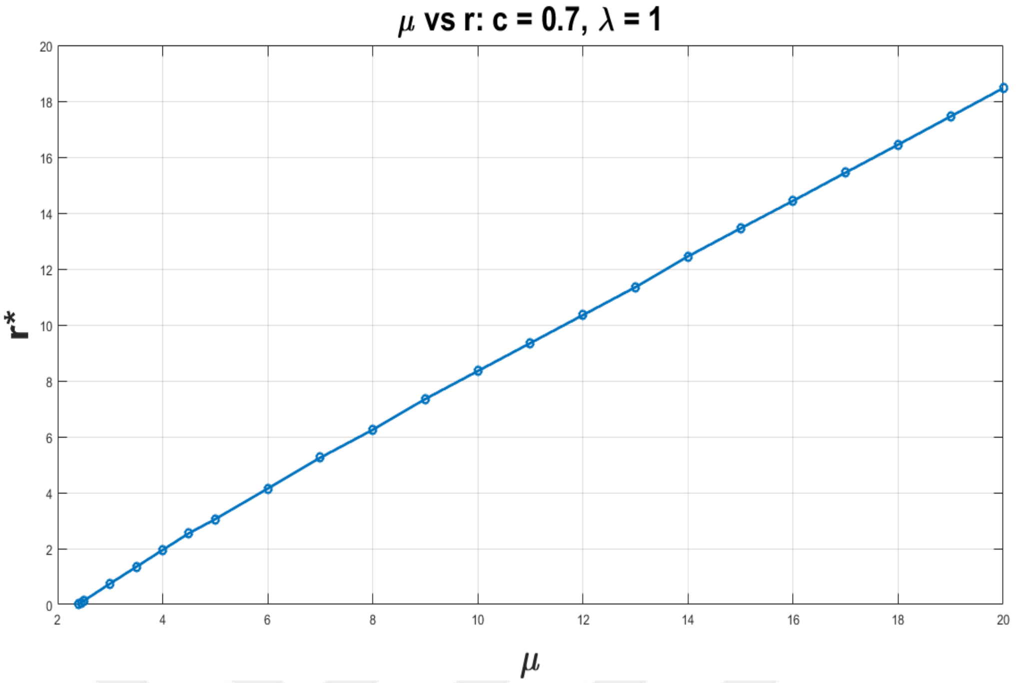


Figure 4.1 : $\mu \quad vs \quad r^*$

Observe that the graph in Figure 4.1 is almost linear. Therefore, a linear regression line was fit to these 23 data points along with calculated value of $r$, which can be seen in Figure 4.2 below:

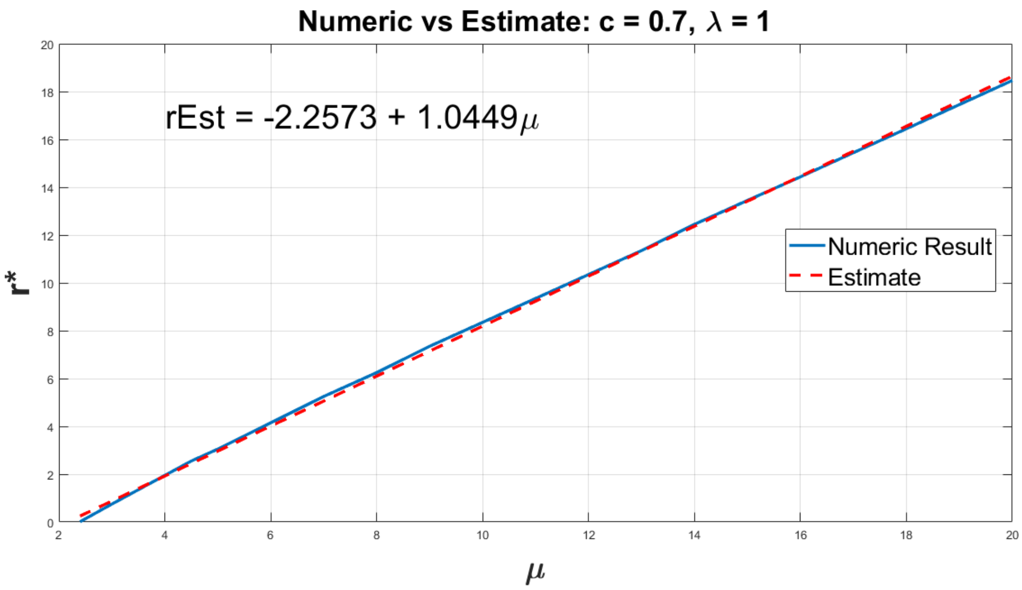


Figure 4.2 : Numeric vs Regression Estimate of r

Linear regression equation is obtained in MATLAB as follows, with Least Square Estimation (LSE) method:

$$\hat{r} = -2.2573 + 1.0449 \cdot \mu + \varepsilon, \quad for\, \mu \geq 2.396$$

in which, the intercept is $-2.2573$ and the slope is $1.0449$. Here, $\varepsilon$ is a noise factor.

In Table 4.8 below, an average absolute error is calculated for the non-extreme errors.

Table 4.8 : Summary table for special case 1

| n | μ | r | $\hat{r}$ | Abs. Err. |
|---|---|---|---|---|
| 1 | 3.5 | 1.35 | 1.3999 | **3.69%** |
| 2 | 4 | 1.95 | 1.9223 | **1.42%** |
| 3 | 4.5 | 2.55 | 2.4448 | **4.13%** |
| 4 | 5 | 3.05 | 2.9672 | **2.71%** |
| 5 | 6 | 4.15 | 4.0121 | **3.32%** |
| 6 | 7 | 5.25 | 5.0570 | **3.68%** |
| 7 | 8 | 6.25 | 6.1019 | **2.37%** |
| 8 | 9 | 7.35 | 7.1468 | **2.76%** |
| 9 | 10 | 8.35 | 8.1917 | **1.90%** |
| 10 | 11 | 9.35 | 9.2366 | **1.21%** |
| 11 | 12 | 10.35 | 10.2815 | **0.66%** |
| 12 | 13 | 11.35 | 11.3264 | **0.21%** |
| 13 | 14 | 12.45 | 12.3713 | **0.63%** |
| 14 | 15 | 13.45 | 13.4162 | **0.25%** |
| 15 | 16 | 14.435 | 14.4611 | **0.18%** |
| 16 | 17 | 15.445 | 15.5060 | **0.39%** |
| 17 | 18 | 16.445 | 16.5509 | **0.64%** |
| 18 | 19 | 17.455 | 17.5958 | **0.81%** |
| 19 | 20 | 18.465 | 18.6407 | **0.95%** |
| **AVG:** | | | | **1.68044%** |

Previously, we kept $c = 0.7$ constant and varied $\mu$ values, until NPC condition was violated. The similar analysis was done for $c = 0.5$ and $c = 0.3$, whose plots are drawn along with when $c = 0.7$. As one can see, $\mu$ vs $r^*$ values are almost linearly related for different values of $c$, in each case with its NPC condition cutt off value.

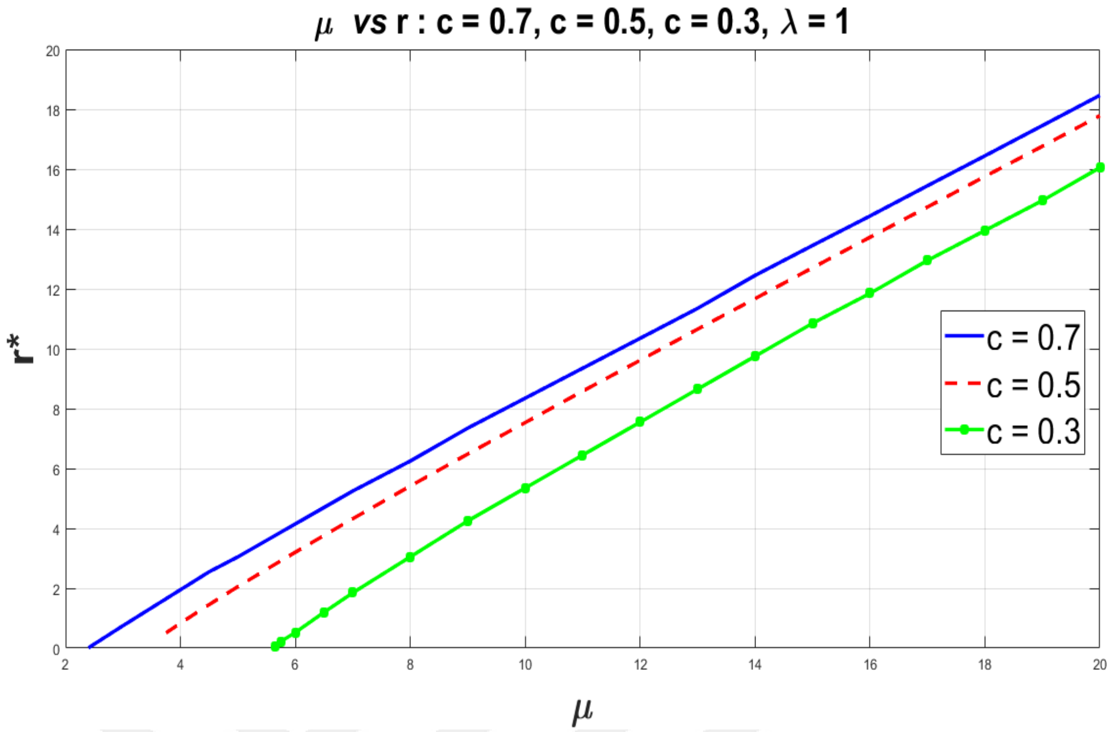


Figure 4.3 : All three cases; c = 0.3, 0.5 , 0.7

**Conclusion – 1.1:** To find a Lundberg-type upper bound for the ruin probability for the special case 1 when $X_1 \sim Exp(\mu)$ and $W_1 \sim Exp(\lambda)$, instead of calculating an integral equation in (4.1), we could simply approximate $r$ with a linear function, without much loss of a precision.

### 4.3.2 Special Case 2: Exponential Claim Arrivals and Uniform Claim Size Distributions

In section 4.1, we have already established the following equation for Special Case 1 when : $X_1 \sim U[0, b]$ and $W_1 \sim Exp(\lambda)$, where $\lambda = 1$ and $c = 0.7$

$$f_1(r) = \int_0^{\infty} \exp(-t) \frac{dt}{(1+t)^{0.7 \cdot r}} = \frac{br}{e^{br} - 1} = f_2(r) \tag{4.9}$$

By applying the similar procedure as in special case 1, we obtain the following values for $r$ in table 4.9 below:

Table 4.9: Summary table for special case 2

| n | b | r | | Err. | n | b | r | $\hat{r}$ | Err. |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 0.025 | 292.5 | 292.1082 | 0.13% | **16** | 0.6 | 2.25 | 1.3213 | 41.28% |
| **2** | 0.05 | 126.5 | 129.1308 | 2.08% | **17** | 0.425 | 5.75 | 4.5366 | 21.10% |
| **3** | 0.075 | 77 | 73.2458 | 4.88% | **18** | 0.45 | 5.05 | 3.8036 | 24.68% |
| **4** | 0.1 | 52.5 | 50.8184 | 3.20% | **19** | 0.475 | 4.45 | 3.1891 | 28.34% |
| **5** | 0.125 | 39.25 | 39.4034 | 0.39% | **20** | 0.5 | 3.95 | 2.6738 | 32.31% |
| **6** | 0.15 | 30.75 | 32.0678 | 4.29% | **21** | 0.525 | 3.45 | 2.2418 | 35.02% |
| **7** | 0.175 | 24.65 | 26.5934 | 7.88% | **22** | 0.55 | 3.05 | 1.8796 | 38.37% |
| **8** | 0.2 | 20.35 | 22.2080 | 9.13% | **23** | 0.575 | 2.65 | 1.5759 | 40.53% |
| **9** | 0.225 | 17.2 | 18.5930 | 8.10% | **24** | 0.6 | 2.25 | 1.3213 | 41.28% |
| **10** | 0.25 | 14.6 | 15.5808 | 6.72% | **25** | 0.65 | 1.65 | 0.9288 | 43.71% |
| **11** | 0.275 | 12.7 | 13.0610 | 2.84% | **26** | 0.7 | 1.15 | 0.6529 | 43.22% |
| **12** | 0.3 | 10.9 | 10.9500 | 0.46% | **27** | 0.75 | 0.675 | 0.4590 | 32.00% |
| **13** | 0.325 | 9.55 | 9.1805 | 3.87% | **28** | 0.775 | 0.425 | 0.3848 | 9.45% |
| **14** | 0.35 | 8.35 | 7.6972 | 7.82% | **29** | 0.8 | 0.275 | 0.3226 | 17.33% |
| **15** | 0.375 | 7.35 | 6.4535 | 12.20% | | | | | |
| | | | | | **AVG:** | | | | **18.00%** |

And the plot of the obtained values for $r$ for given values of $b$ is given in Figure 4.4 below.

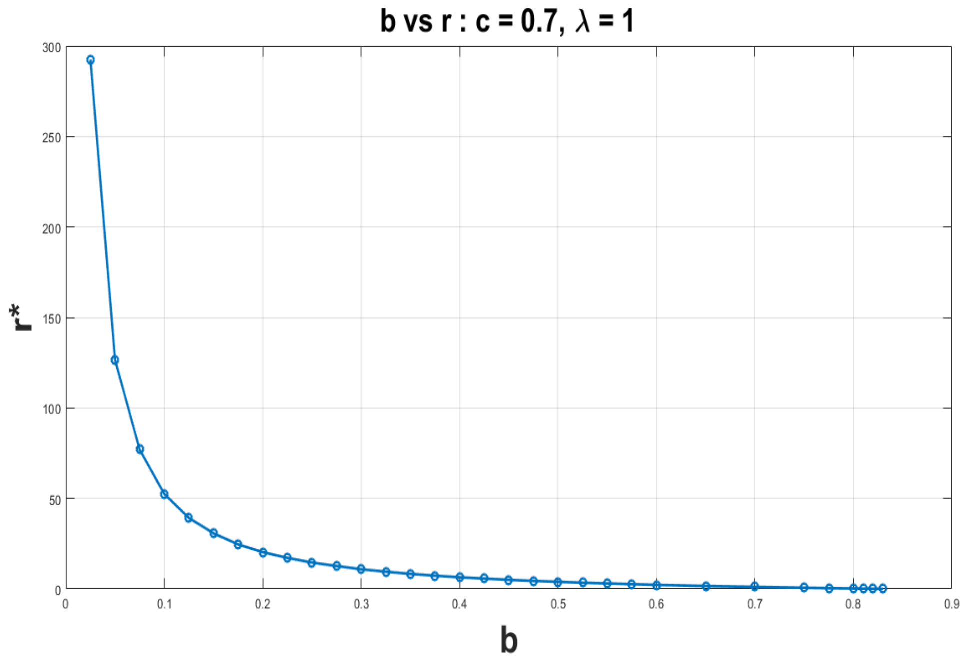


Figure 4.4: $b \quad vs \quad r^*$

Similarly, Exponential Regression Fit is utilized, and the following equation is obtained in MATLAB

$$\hat{r} = 714.2 \cdot e^{-47.83 \cdot b} + 90.74 \cdot e^{-7.049 \cdot b} + \varepsilon, \qquad for\ 0 < b \leq 0.83482$$

In Figure 4.5 below, the plot of the numeric result and the plot of the estimate is given.

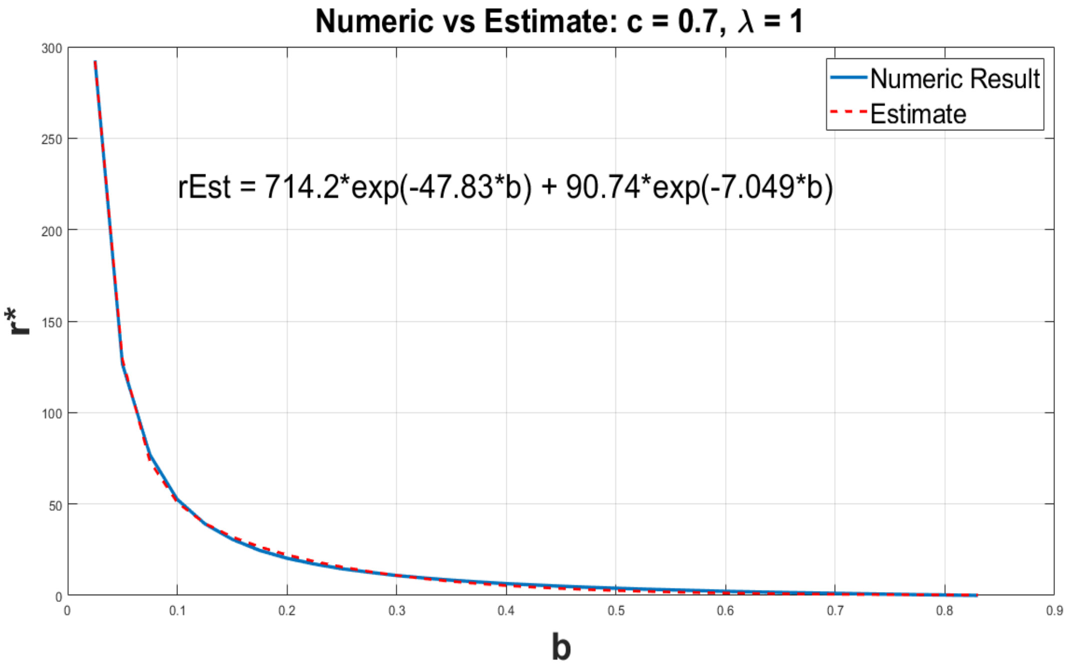


Figure 4.5: Numeric vs Exponential Estimate of $r$

Question : Why does solution cut off at around $b \approx 0.83$ ?

Answer : Net Profit Condition!

- We know that $E(X_1) = \frac{b}{2} < cE(g(W_1))$ must be satisfied.
- Here, $c = 0.7$
- Also, MATLAB calculation yields : $E(g(W_1)) \approx 0.5963$

Solving for $b$, we find that $b \approx 2 \cdot 0.7 \cdot 0.5963 = 0.83482$, as expected.

Again, when values of $c$ are 0.7, 0.5 and 0.3, the plot of the solution for $r$ for given values of $b$ are given in Figure 4.6 below.

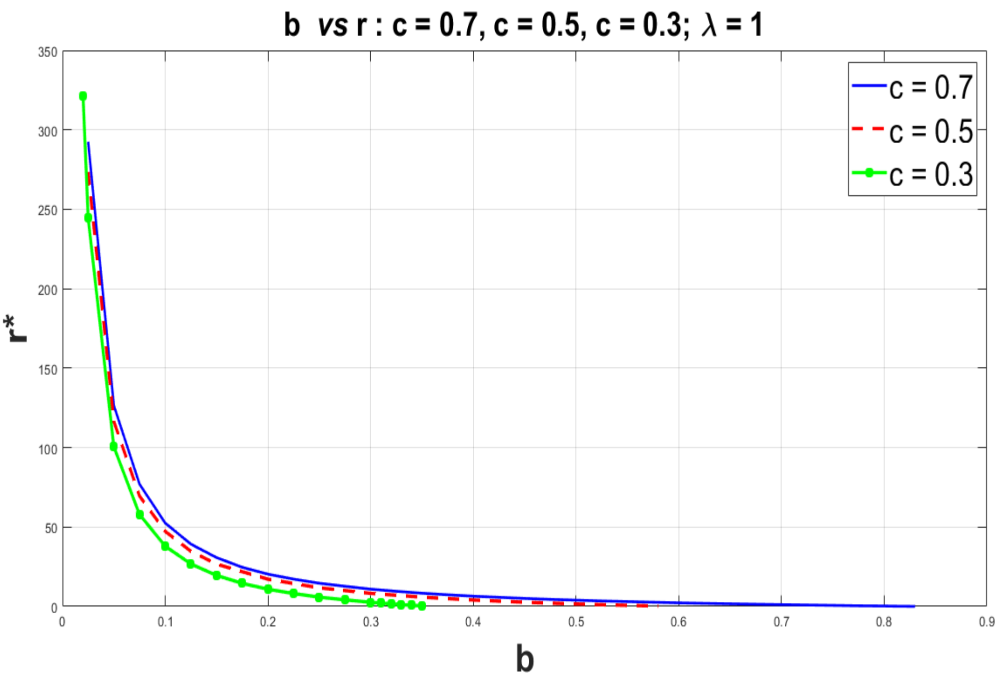


Figure 4.6: All three cases; c = 0.3, 0.5 , 0.7

**Conclusion – 1.2:** To find a Lundberg type upper bound for the ruin probability, instead of calculating an integral equation in (4.3), we could simply approximate $r$ with an exponential function. Observe that the absolute error in this case is relatively higher. Thats attibuted to the high sensitivity of the model fitted.

## 5. APPROXIMATION TO CRAMÉR TYPE BOUND FOR NON-LINEAR MODEL

In part 4 of the thesis, we only found a loose upper bound for our non-linear model. In this part of the thesis, we try to find tighter bound for the ruin probability. More specifically, we find approximate upper and lower bounds for the ruin probability for the non-linear model, which is known as Cramér-type bound in the literature.

In fact, in Theorem 5.1 below, a neat formula and expression is given for calculating such bounds. However, it is rather too complicated integral equation. Additionally, it is very hard to interpret it. In fact, for only a few distributions the ruin probability $\psi(u)$ can be expressed as an explicit function of the ingredients of the risk process. Most of the time this requires numeric methods or Monte Carlo approximations to $\psi(u)$.

What we did in in this work is, we exploited the statistical characteristics of the random variable, $\hat{X}$, which describes the residual time (limit distribution) of the renewal process produced by the sequence $\{X_n\}$, representing the accidents(damages). In particular, moment generating function of $\hat{X}$ was utilized to determine a constant $C$, which is an unknown coefficient in the bound expression of the ruin probability as can be seen in Theorem 5.1. In order to simplify these expressions and transform them into a compact form, calculus methods were used.

First, let us present one of the most important results of risk theory, thanks to Cramér [10].

**Theorem 5.1** (Cramer’s ruin bound)

Consider the Cramér-Lundberg model which satisfies the net profit condition. Additionally, assume that the claim size distribution function $F_{X_1}$ has a density, the moment generating function of $X_1$ exists in some neighborhood $(-h_0, h_0)$ of the origin, the adjustment coefficient $r$ exists and lies in $(0, h_0)$. Then there exists a constant $\mathrm{C} > 0$ such that

$$\lim_{u \to \infty} e^{ru}\psi(u) = C$$

and

$$C = \left[ \frac{r}{\rho E(X_1)} \int_0^\infty x e^{rx} \bar{F}_{X_1}(x) dx \right]^{-1}$$

For proof of the Theorem 4.1, please refer to Mikosch (2004)

The reciprocal expression for the constant $C$ is given in (5.1) below. To compute this $C$, one needs to know the adjustment coefficient $r$, the expected claim size $EX_1$ , the safety loading $\rho$ and some other characteristics of $F_{X_1}$ as well. To satisfy net profit condition (NPC), we have chosen $\rho$ to be as follows, i.e.,

$$\rho = c \frac{E[g(W_1)]}{EX_1} - 1 > 0$$

Please observe that it is not an easy task to calculate the value of $C$, at least it is not a straightforward task. Now let us rewrite the expression for $C$ as in equation (5.1) below and try to simplify it by expanding it:

$$\frac{1}{C} = \frac{r}{\rho E X_1} \int_0^\infty x e^{rx} \bar{F}_{X_1} dx \tag{5.1}$$

Before we move on, please note that $F_A$ denotes the distribution function of any random variable $A$ and accordingly $\bar{F}_A = 1 - F_A$ denotes its tail.

Let's denote the integration part as:

$$\boldsymbol{I}(\boldsymbol{r}) \equiv \int_0^\infty x e^{rx} \bar{F}_{X_1}(x) dx \tag{5.2}$$

for ease of computability.

From calculus, we use the following fact, Taylor Series Expansion:

$$e^{rx} = \sum\nolimits_{n=0}^{\infty} \frac{(rx)^n}{n!} \tag{5.3}$$

After inserting Taylor form of $e^{rx}$ in (5.3) into (5.2) above, we get the following:

$$\boldsymbol{I}(\boldsymbol{r}) \equiv \int_0^\infty x\left(\sum_{n=0}^\infty \frac{(rx)^n}{n!}\right)\bar{F}_{X_1}(x)dx = \sum_{n=0}^\infty \frac{(r)^n}{n!}\int_0^\infty x^{n+1}\,\bar{F}_{X_1}(x)dx \qquad (5.4)$$

Another well-known fact from statistics is the following identity:

$$E(X^k) \equiv k\int_0^\infty x^{k-1}\,\bar{F}_{X_1}(x)dx \qquad (5.5)$$

Leaving the integration part of (5.5) on the left-hand side, we have the following equation:

$$\int_0^\infty x^{n+1}\,\bar{F}_{X_1}(x)dx = \frac{1}{n+2}E(X^{n+2}) \qquad (5.6)$$

Then inserting (5.6) into (5.4) we get:

$$\boldsymbol{I}(\boldsymbol{r}) = \sum_{n=0}^\infty \frac{r^n}{n!}\,\frac{1}{n+2}E(X^{n+2}) \qquad (5.7)$$

Again, for ease of notation, let $m_k \equiv E(X^k)\ \ for\ k = 1, 2$ ...
Then $I(r)$ in (5.7) can be rewritten as follows:

$$\boldsymbol{I}(\boldsymbol{r}) = \sum_{n=0}^\infty \frac{(n+1)r^n}{(n+2)!}\,m_{n+2} \qquad (5.8)$$

By change of index in (5.8) as $n = i - 2$, we get the following:

$$\begin{aligned}\boldsymbol{I}(\boldsymbol{r}) &= \sum_{i=2}^\infty \frac{(i-1)r^{i-2}}{i!}\,m_i = \sum_{i=2}^\infty \frac{ir^{i-2}}{i!}\,m_i - \sum_{i=2}^\infty \frac{r^{i-2}}{i!}\,m_i\ = \\ &= \sum_{i=2}^\infty \frac{r^{i-2}}{(i-1)!}\,m_i - \sum_{i=2}^\infty \frac{r^{i-2}}{i!}\,m_i \\ &= \frac{1}{r}\sum_{i=2}^\infty \frac{r^{i-1}}{(i-1)!}\,m_i - \frac{1}{r^2}\sum_{i=2}^\infty \frac{r^i}{i!}\,m_i \\ &= \frac{1}{r}\sum_{j=1}^\infty \frac{r^j}{j!}\,m_{j+1} - \frac{1}{r^2}\sum_{j=2}^\infty \frac{r^j}{j!}\,m_j\end{aligned}$$

$$= m_2 + \frac{1}{r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\,m_{j+1} - \frac{1}{r^2}\sum_{j=2}^{\infty}\frac{r^j}{j!}\,m_j$$

$$= m_2 + \sum_{j=2}^{\infty}\frac{r^j}{j!}\,\frac{m_{j+1}}{r} - \sum_{j=2}^{\infty}\frac{r^j}{j!}\,\frac{m_j}{r^2} = m_2 + \sum_{j=2}^{\infty}\left[\frac{r^j}{j!}\frac{m_{j+1}}{r} - \frac{r^j}{j!}\frac{m_j}{r^2}\right]$$

$$= m_2 + \sum_{j=2}^{\infty}\frac{r^j}{j!}\left[\frac{m_{j+1}}{r} - \frac{m_j}{r^2}\right] = m_2 + \frac{1}{r^2}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[rm_{j+1} - m_j\right]$$

In summary,

$$\boldsymbol{I}(\boldsymbol{r}) = m_2 + \frac{1}{r^2}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[rm_{j+1} - m_j\right] \tag{5.9}$$

And also, since $\boldsymbol{I}(\boldsymbol{r}) \equiv \int_0^{\infty} xe^{rx}\,\bar{F}_{X_1}(x)dx$ from (5.2) above, we have

$$\boldsymbol{I}(\boldsymbol{r}) = m_2 + \frac{1}{r^2}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[rm_{j+1} - m_j\right] = \int_0^{\infty} xe^{rx}\,\bar{F}_{X_1}(x)dx = \boldsymbol{I}(\boldsymbol{r}) \tag{5.10}$$

Now we can rewrite the starting equation in (5.1) as:

$$\frac{1}{C} = \frac{r}{\rho EX_1}\int_0^{\infty} xe^{rx}\,\bar{F}_{X_1}(x)dx = \frac{r}{\rho m_1}\left\{m_2 + \frac{1}{r^2}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[rm_{j+1} - m_j\right]\right\} \tag{5.11}$$

Let us simplify (5.11) even further below:

$$\frac{1}{C} = \frac{r}{\rho m_1}\left\{m_2 + \frac{1}{r^2}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[rm_{j+1} - m_j\right]\right\}$$

$$= \frac{rm_2}{\rho m_1} + \frac{1}{\rho m_1 r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[rm_{j+1} - m_j\right]$$

$$= \frac{2rm_2}{\rho 2m_1} + \frac{1}{\rho m_1 r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[rm_{j+1} - m_j\right]$$

In summary,

$$\frac{1}{C} = \frac{2rm_2}{\rho 2m_1} + \frac{1}{\rho m_1 r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[rm_{j+1} - m_j\right] \tag{5.12}$$

### 5.1 Expressing the constant $C$ in terms of $M_{\hat{X}}(r)$ and $M'_{\hat{X}}(r)$

Similarly, for simplicity, let's denote the random variable which describes the residual time (limit distribution) of the renewal process produced by the sequence $\{X_n\}$ with $\hat{X}$. Then, $\widehat{m}_j's$ will become the $j^{th}$ moment of the limit distribution (of residual time) generated by the $\{X_n\}$ sequence of renewal process.

Then, we can write the following well-known identity from renewal theory:

$$\frac{m_{j+1}}{(j+1)m_1} \equiv \widehat{m}_j$$
$$m_{j+1} \equiv (j+1)m_1\widehat{m}_j$$
$$m_j \equiv jm_1\widehat{m}_{j-1}$$

In this case, the MGF of the random variable $\hat{X}$ can be expresses as follows:

$$M_{\hat{X}}(r) \equiv E\left(e^{r\hat{X}}\right) = \sum_{j=0}^{\infty}\frac{r^j}{j!}\widehat{m}_j = 1 + \sum_{j=1}^{\infty}\frac{r^j}{j!}\widehat{m}_j$$

Re-arranging the terms in (5.12), and expressing it in terms of $\widehat{m}_j$'s we get:

$$\frac{1}{C} = \frac{2rm_2}{\rho 2m_1} + \frac{1}{\rho m_1 r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[r(j+1)m_1\widehat{m}_j - jm_1\widehat{m}_{j-1}\right]$$
$$= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[r(j+1)\widehat{m}_j - j\widehat{m}_{j-1}\right]$$

In summary,

$$\frac{1}{C} = \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[r(j+1)\widehat{m}_j - j\widehat{m}_{j-1}\right] \tag{5.13}$$

Here, note that

$$M_{\hat{X}}(r) \equiv E\left(e^{r\hat{X}}\right) = \sum_{j=0}^{\infty}\frac{r^j}{j!}\widehat{m}_j = 1 + \sum_{j=1}^{\infty}\frac{r^j}{j!}\widehat{m}_j$$

Thus,

$$\sum_{j=1}^{\infty}\frac{r^j}{j!}\widehat{m}_j \ = E\left(e^{r\hat{X}}\right) - 1 \tag{5.14}$$

Similarly,

$$\begin{aligned} M'_{\hat{X}}(r) \equiv E\left(\hat{X}e^{r\hat{X}}\right) &= \sum_{j=1}^{\infty}\frac{jr^{j-1}}{j!}\widehat{m}_j = \sum_{j=1}^{\infty}\frac{r^{j-1}}{(j-1)!}\widehat{m}_j \\ &= \sum_{k=0}^{\infty}\frac{r^k}{k!}\widehat{m}_{k+1} = \widehat{m}_1 + \sum_{j=1}^{\infty}\frac{r^j}{j!}\widehat{m}_{j+1} \end{aligned}$$

Thus,

$$\sum_{j=1}^{\infty}\frac{r^j}{j!}\widehat{m}_{j+1} \ = \ E\left(\hat{X}e^{r\hat{X}}\right) - \widehat{m}_1 = M'_{\hat{X}}(r) - \widehat{m}_1 \tag{5.15}$$

Now we can express ${}^1\!/_C$ in term of $M_{\hat{X}}(r)$ and $M'_{\hat{X}}(r)$ by expanding (5.13),

$$\begin{aligned} \frac{1}{C} &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[r(j+1)\widehat{m}_j - j\widehat{m}_{j-1}\right] \\ &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\sum_{j=2}^{\infty}\frac{r^j}{j!}\left[j\widehat{m}_j r + \widehat{m}_j r - j\widehat{m}_{j-1}\right] \\ &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\left[\sum_{j=2}^{\infty}\frac{r^j}{j!}j\widehat{m}_j r + \sum_{j=2}^{\infty}\frac{r^j}{j!}\widehat{m}_j r - \sum_{j=2}^{\infty}\frac{r^j}{j!}j\widehat{m}_{j-1}\right] \\ &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\left[\sum_{j=2}^{\infty}\frac{r^{j+1}}{(j-1)!}\widehat{m}_j + \sum_{j=2}^{\infty}\frac{r^{j+1}}{j!}\widehat{m}_j - \sum_{j=2}^{\infty}\frac{r^j}{(j-1)!}\widehat{m}_{j-1}\right] \\ &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\left[\sum_{k=1}^{\infty}\frac{r^{k+2}}{k!}\widehat{m}_{k+1} + \sum_{k=1}^{\infty}\frac{r^{k+1}}{k!}\widehat{m}_k - r^2\widehat{m}_1 - \sum_{k=1}^{\infty}\frac{r^{k+1}}{k!}\widehat{m}_k\right] \\ &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\left[r^2\sum_{k=1}^{\infty}\frac{r^k}{k!}\widehat{m}_{k+1} + r\sum_{k=1}^{\infty}\frac{r^k}{k!}\widehat{m}_k - r^2\widehat{m}_1 - r\sum_{k=1}^{\infty}\frac{r^k}{k!}\widehat{m}_k\right] \\ &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\left[r^2(M'_{\hat{X}}(r) - \widehat{m}_1) + r(M_{\hat{X}}(r) - 1) - r^2\widehat{m}_1 - r(M_{\hat{X}}(r) - 1)\right] \\ &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}\left[r^2 M'_{\hat{X}}(r) - 2r^2\widehat{m}_1\right] \\ &= \frac{2r}{\rho}\widehat{m}_1 + \frac{1}{\rho r}r^2 M'_{\hat{X}}(r) - \frac{2}{\rho r}r^2\widehat{m}_1 \end{aligned}$$

$$= \frac{2r}{\rho}\widehat{m}_1 + \frac{r}{\rho}M'_{\hat{X}}(r) - \frac{2r}{\rho}\widehat{m}_1$$

$$= \frac{r}{\rho}M'_{\hat{X}}(r)$$

We finally arrive at the following compact and easy to interpret result,

$$\boldsymbol{\frac{1}{C} = \frac{r}{\rho}M'_{\hat{X}}(r)} \tag{5.16}$$

Here, $F(x)$ denotes the claim size distribution of $X_1, X_2 \dots$ and $m_1 = E(X_1)$. $\hat{X}$ is the random variable describing the residual time (limit distribution) of the renewal process produced by the sequence $\{X_n\}$ and its c.d.f is given as follows:

$\boldsymbol{F_{\hat{X}}(x) \equiv P\{\hat{X} \le x\} = \frac{1}{m_1}\int_0^x (1 - F(x))dx}$ (*Smith's Key Renewal Theorem*)

Thus, $f_{\hat{X}}(x) = F'_{\hat{X}}(x) = \frac{1}{m_1}(1 - F(x))$ is p.d.f. of $\hat{X}$ .

Here, $M_{\hat{X}}(r) = E(e^{r\hat{X}}) = \int_0^\infty e^{rx} f_{\hat{X}}(x)\,dx$ and $M'_{\hat{X}}(r) = E(\hat{X}e^{r\hat{X}})$

## 5.2 Numerical calculation of $C$ for some special cases

The main purpose of the thesis is to calculate ruin probabilities of the non-linear model. In (5.16), we have derived a compact formula in regard to Theorem 5.1 for calculation of a constant $C$. However, it is not possible to calculate $C$ for an arbitrary distribution. Therefore, we resort to special cases below.

**Special Case – 1: $\boldsymbol{X_1 \sim}$ Exp($\boldsymbol{\mu}$), $\boldsymbol{\mu > 0}$ and $\boldsymbol{W_1 \sim Exp(\lambda), \lambda > 0}$**

Let $X_1 \sim$ Exp($\mu$), $\mu > 0$ and $W_1 \sim Exp(\lambda), \lambda > 0$

***1.*** *CDF of* $\boldsymbol{X_1}$: $F(x) = 1 - e^{-\mu x}$, $x \in [0, \infty)$, $\mu > 0$

***2.*** $f_{\hat{X}}(x) = \frac{1}{m_1}\big(1 - F(x)\big) = \mu\big(1 - (1 - e^{-\mu x})\big) = \mu e^{-\mu x}$ $\{= f_{X_1}(x)\}$

***3.*** $M_{\hat{X}}(r) = E(e^{r\hat{X}}) = \int_0^\infty e^{rx} f_{\hat{X}}(x)dx =$

$$= \int_0^\infty e^{rx}\frac{1}{m_1}\big(1 - F(x)\big)dx = \int_0^\infty e^{rx}\,\mu e^{-\mu x}dx$$

$$= \mu \int_0^\infty e^{-(\mu-r)x}\, dx = \frac{\mu}{\mu - r}, \qquad r < \mu$$

**4.** $M'_{\hat{X}}(r) = E(\hat{X}e^{r\hat{X}}) = \left(\mu \int_0^\infty e^{(r-\mu)x}\, dx\right)'_r = (\frac{\mu}{\mu-r})'_r = \frac{\boldsymbol{\mu}}{(\boldsymbol{\mu}-\boldsymbol{r})^2}$

$\rho = c\frac{\boldsymbol{E}[\boldsymbol{g}(\boldsymbol{W_1})]}{\boldsymbol{EX_1}} - 1 > 0$ < Safety loading, $c = (1+\rho)\frac{EX_1}{EW_1}$ >

$E(g(W_1)) \approx g\big(E(W_1)\big) + \frac{g''(E(W_1))}{2!}Var(W_1)$ <Taylor expansion>

$a = EW_1 = \frac{1}{\lambda}, \lambda > 0\ ; m_1 = E(X_1) = \frac{1}{\mu}$

$Var(W_1) = \frac{1}{\lambda^2}$

$g(x) = \ln(1+x)\ ; g'(x) = \frac{1}{1+x}\ ; g''(x) = \frac{-1}{(1+x)^2}$

Let $\lambda = 1, c = 0.7, \mu = 10$

Table 5.1 : The value of $r$ for $\lambda = 1; c = 0.7; \mu = 10$

| **λ=1 ; c=0.7 ; μ=10 ;** | | | | |
|---|---|---|---|---|
| **n** | **r** | **$f_1(r)$** | **$f_2(r)$** | **$f_1(r) - f_2(r)$** |
| **0** | 8.0000 | 0.172479 | 0.200000 | -0.027521 |
| **1** | 8.1000 | 0.170494 | 0.190000 | -0.019506 |
| **2** | 8.2000 | 0.168551 | 0.180000 | -0.011449 |
| **3** | **8.3000** | 0.166651 | 0.170000 | **-0.003349** |
| **4** | **8.4000** | 0.164792 | 0.160000 | **0.004792** |
| **5** | 8.5000 | 0.162973 | 0.150000 | 0.012973 |
| **6** | 8.6000 | 0.161192 | 0.140000 | 0.021192 |
| **7** | 8.7000 | 0.159449 | 0.130000 | 0.029449 |
| **8** | 8.8000 | 0.157741 | 0.120000 | 0.037741 |
| **9** | 8.9000 | 0.156069 | 0.110000 | 0.046069 |
| **10** | 9.0000 | 0.154430 | 0.100000 | 0.054430 |
| $\boldsymbol{r^*} \approx \boldsymbol{8.35}$ ; $\boldsymbol{Ruin\ Pr}$: $\boldsymbol{\psi(u) \leq e^{-8.35\cdot u}}$ | | | | |

$$f_1(r) = \int_0^\infty \frac{e^{-t}}{(1+t)^{0.7r}}\,\mathrm{dt} = 1 - \frac{r}{\mu} = f_2(r), \quad for\ \ r > 0$$

$\Rightarrow r \approx 8.35$

- $\rho = c\mu E[g(W_1)] - 1 \approx c\mu\left(g(a) + \frac{g''(a)}{2!}Var(W_1)\right) - 1 =$

$$= 0.7 * 10\left(\ln\left(1 + \frac{1}{1}\right) + \frac{-1}{2\left(1 + \frac{1}{1}\right)^2}\frac{1}{1^2}\right) - 1 =$$

$$= 7\left(\ln(2) - \frac{1}{8}\right) - 1 \approx 2.98$$

- $M'_{\hat{X}}(r) = E\left(\hat{X}e^{r\hat{X}}\right) = \frac{\mu}{(\mu-r)^2} = \frac{10}{(10-8.35)^2} = \frac{10}{2.7225}$

$$\frac{1}{C} = \frac{r}{\rho}M'_{\hat{X}}(r) \Rightarrow \boldsymbol{C} = \frac{\rho}{r * M'_{\hat{X}}(r)} = \frac{2.98}{8.35}\frac{2.7225}{10} \approx \mathbf{0.098}$$

$$\Rightarrow \boldsymbol{\psi(u)} \approx Ce^{-ru} = \mathbf{0.098}\boldsymbol{e^{-8.35*u}}$$

Here, $u$ denotes the initial capital.

Now, lets confirm this above calculation by the formula given in Theorem 5.1, i.e.,

$$C = \left[\frac{r}{\rho E(X_1)}\int_0^\infty xe^{rx}\bar{F}_{X_1}(x)dx\right]^{-1}$$

We already know that,

$r \approx 8.35$

$\rho \approx 2.98$

$E(X_1) = \frac{1}{\mu} = \frac{1}{10}$

and calculating the integral below:

$$\int_0^\infty xe^{rx}\bar{F}_{X_1}(x)dx \approx \int_0^\infty xe^{8.35x}\left(1 - (1 - e^{-\mu x})\right)dx = \int_0^\infty xe^{(8.35-\mu)x}dx$$

$$= \int_0^\infty xe^{(8.35-10)x}dx = \int_0^\infty xe^{-1.65x}dx \approx 0.3673$$

Thus, inserting the values we get:

$$C \approx \left[\frac{8.35}{2.98 * \frac{1}{10}} * 0.3673\right]^{-1} = \left[\frac{83.5 * 0.3673}{2.98}\right]^{-1} = 0.0972$$

Which confirms the approximate result found alternatively.

**Conclusion – 2:** Instead of calculating,

$$C = \left[\frac{r}{\rho E(X_1)} \int_0^\infty x e^{rx} \bar{F}_{X_1}(x) dx \right]^{-1}$$

we have found an alternative way of computing the value of $C$ as follows:

$$\frac{1}{C} = \frac{r}{\rho} M'_{\hat{X}}(r)$$

- $\hat{X}$ is a random variable describing the residual time (limit distribution) of the renewal process produced by the sequence $\{X_n\}$ which represents the accidents(damages)
- $M_{\hat{X}}(r)$ is the moment generating function of $\hat{X}$

# 6. CONCLUSION

In this work, a special non-linear risk process was constructed and studied. This model extends the classical risk process by relaxing the assumption that the premium income function is linear in time, $p(t) = ct$. In our model, we let the premium income function $g(t) = \ln(1 + t)$ to be logarithmic, although it could be any general function whose rate of growth decreases with time while it is still monotonically increasing. We called this stochastic process as *Logarithmic Risk Process*, $V(t)$. In the first part of the thesis, we found a Lundberg-type upper bound of ruin probability for this non-linear process. While trying to calculate this bound, non-linear equations were encountered. To solve these non-linear equations, numerical methods were employed. In the second part of the study, a Cramér-type bound of ruin probability was calculated for this model. Statistical characteristics of the random variable, $\hat{X}$ , which denotes the residual time (limit distribution) of the renewal process produced by the sequence $\{X_n\}$, representing the accidents(damages), was exploited. In particular, moment generating function of $\hat{X}$ was utilized to determine a constant $C$, which is an unknown coefficient in the bound expression of the ruin probability. In order to simplify these expressions and transform them into a compact form, calculus methods were used. Also, results from part one was used in conjunction with results in part two while finding approximate value for the value of $C$. In both parts, various scenarios, in a sense of distribution functions, were considered in order to investigate and calculate ruin probabilities.

## REFERENCES

[1] Asmussen S. (2000), Ruin Probabilities. Singapore, World Scientific.

[2] Asmussen, S., Rolski, T., (1994), “Risk Theory in a Periodic Environment: The Cramér-Lundberg Approximation and Lundberg's Inequality”, Mathematics of Operations Research , May 1994, Vol. 19, No. 2, pp. 410-433

[3] Avram, F., Biard, R., Dutang, C., Loisel, S., Rabehasaina, L., (2014), A survey of some recent results on Risk Theory. ESAIM: Proceedings, EDP Sciences, 2014, 44, pp.322 – 337

[4] Boikov, A.V., The Cramér-Lundberg model with stochastic premium process, Theory of Probabilistic Applications 47 (2002) 489-493.

[5] Bühlmann H. (1970), Mathematical Methods in Risk Theory. Sipringer, Berlin.

[6] Chadjiconstantinidis, S., Politis, K., (2007), "Two-sided bounds for the distribution of the deficit at ruin in the renewal risk model", Insurance: Mathematics and Economics Volume 41, Issue 1, Pages 41-52

[7] Cohen, A., R.Young, V., (2020), “Rate of convergence of the probability of ruin in the Cramér–Lundberg model to its diffusion approximation”, Insurance: Mathematics and Economics, Volume 93, Pages 333-340

[8] Constantinescu, C., Samorodnitsky, G., Zhu, W., (2018), “Ruin probabilities in classical risk models with gamma claims”, Scandinavian Actuarial Journal, Volume 2018, 2018 - Issue 7

[9] Cossettea, H., Marceaua, E., Trufinb, J., Zuyderhoffb, P., (2020), "Ruin-based risk measures in discrete-time risk models", Insurance: Mathematics and Economics Volume 93, Pages 246-261

[10] Cramér, H., (1930), “On the mathematical theory of risk”, Skandia Jubilee Volume, Stockholm. Reprinted in: martin-Löf, A. (Ed.) Cramér, H. (1994) Collected Works. Springer, Pages 155-166

[11] Dickson, D.C.M., (2012), "The joint distribution of the time to ruin and the number of claims until ruin in the classical risk model", Insurance: Mathematics and Economics Volume 50, Issue 3, Pages 334-337

[12] Embrechts P., Klüppelberg C., Mikosch T., (1997), Modelling Extremal Events for Insurance and Finance. Springer, Heidelberg.

[13] Embrechts P., Veraverbeke N., (1982), “Estimates for the Probability of Ruin with Special Emphasis on the Possibility of Large Claims”, Insurance: Math. Econom. 1, 55-72.

[14] Feller W. (1971), An Introduction to Probability Theory and Its Applications II. Wiley, New York.

[15] Gaier, J., Grandits, P., Schachermayer, W., (2003), “Asymptotic Ruin Probabilities and Optimal Investment”, The Annals of Applied Probability Vol. 13, No. 3, pp. 1054-1076

[16] Gauchonab, R., Loisela, S., Rullièrea, J., Trufinc, J., (2020), “Optimal prevention strategies in the classical risk model”, Insurance: Mathematics and Economics Volume 91, Pages 202-208

[17] Gerber, H.U., (1982), "Ruin theory in the linear model", Insurance: Mathematics and Economics Volume 1, Issue 3, Pages 213-217

[18] Gerber, H.U., (1988), “Mathematical fun with ruin theory”, Insurance: Mathematics and Economics Volume 7, Issue 1, Pages 15-23

[19] Hanalioğlu Z., Şimşek M. and Khaniyev T., 2017, “Genelleştirilmiş Lundberg Eşitsizliği Üzerine”, 3. Ulusal Sigorta ve Aktüerya Kongresi, 28 – 29 Eylül, Karabük Üniversitesi, Özetler Kitapçığı, s.8.

[20] Kaas R., Goovaerts M., Dhaene J., Denuit M., (2001). Modern Actuarial Risk Theory. Kluwer, Boston.

[21] Klüppelberg, C., Stadtmüller, U., (1995), “Ruin probabilities in the presence of heavy-tails and interest rates” , Scandinavian Actuarial Journal Volume 1998, Issue 1

[22] Landriault, D., Li, B., Loke, S., Willmot, G.E., Xu, D., (2017), "A note on the convexity of ruin probabilities", Insurance: Mathematics and Economics Volume 74, Pages 1-6

[23] Lefèvre, C., Trufin, J., Zuyderhoff, P., (2017), "Some comparison results for finite-time ruin probabilities in the classical risk model", Insurance: Mathematics and Economics Volume 77, Pages 143-149

[24] Lianga, X., Liangb, Z., R.Young, V., (2020), "Optimal reinsurance under the mean–variance premium principle to minimize the probability of ruin", Insurance: Mathematics and Economics Volume 92, Pages 128-146

[25] Lundberg, F., (1903), “Approximerad framställning av sannolikhetsfunktionen”, Återförsäkring av kollektivrisker. Akad. Afhandling. Almqvist och Wiksell, Uppsala, Pages 7-9

[26] Malinovskii, V.K, (1998), “Non-Poissonian claims' arrivals and calculation of the probability of ruin”, Insurance: Mathematics and Economics Volume 22, Issue 2, 1 June 1998, Pages 123-138

[27] Malinovskii, V.K, (2014), “Improved asymptotic upper bounds on the ruin capital in the Lundberg model of risk”, Insurance: Mathematics and Economics Volume 55, Pages 301-309

[28] Mikosch, T. (2004). Non-life insurance mathematics: An Introduction with Stochastic Processes. Berlin: Springer-Verlag.

[29] Mishura,Y., Perestyuk, M., Ragulina, O., (2014), “Ruin probability in a risk model with variable premium intensity and risky investments”, Opuscula Mathematic, March 2014.

[30] Resnick, S.I., (1992), “Advantures in Stochastic Processes”, Birkhäuser, Boston

[31] Rolski, T., Schmidli, H., Schmidt, V., Teugels, J., (1999), “Stochastic Processes for Insurance and Finance”, Wiley, New York.

[32] Stanford, D.A., Stroiński, K.J., Lee, K., (2000), "Ruin probabilities based at claim instants for some non-Poisson claim processes", Insurance: Mathematics and Economics Volume 26, Issues 2–3, Pages 251-267

[33] Straub E. (1988), Non-Life Insurance Methematics. Springer, New York.

[34] Sundt B. (1999), “An Introduction to Non-Life Insurance Mathematics”, 4th edition, VVW Karlsruhe.

[35] Temnov, G., (2014), “Risk Models with Stochastic Premium and Ruin Probability Estimation”, Journal of Mathematical Sciences volume 196, pages 84–96

[36] Willmot, G.E., Lin, X.S., (2001), “Lundberg Approximations for Compound Distributions with Insurance Applications”, Springer, Berlin.

[37] Yang H. (1998), “Non-exponential Bounds for Ruin Probability with Interest Effect Included”, Scand. Actuarial J. 1, 66-79.

[38] Zhang, Z., Yang, H., (2009), “On a risk model with stochastic premiums income and dependence between income and loss”, Journal of Computational and Applied Mathematics, May 2010, Pages 44-57