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Kantorovich Version of Vector-Valued Shepard Operators

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Abstract: In the present work, in order to approximate integrable vector-valued functions, we study the Kantorovich version of vector-valued Shepard operators. We also display some applications supporting our results by using parametric plots of a surface and a space curve. Finally, we also investigate how nonnegative regular (matrix) summability methods affect the approximation.

Keywords: multivariate approximation; approximation of vector-valued functions; Shepard operators; Kantorovich operators; matrix summability methods; Cesàro summability

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1. Introduction

In the early 1900s, S. Bernstein [1] introduced a family of operators known in the literature as *Bernstein polynomials* in order to approximate continuous functions, which enabled us to give a constructive proof of Weierstrass's fundamental approximation theorem. In 1930, L. V. Kantorovich [2,3] gave a modification of the Bernstein polynomials to approximate not only continuous functions but also integrable functions. Later, this idea was applied to many well-known approximation operators. Such operators are known in the literature as *Kantorovich-type operators*. There are numerous studies in the literature related to Kantorovich operators. Especially in recent years, it has also been shown that these operators have significant advantages in fields such as artificial neural networks, signal and digital image processing, and sampling theory (see, for instance, [4–7]).

In this article, we study the Kantorovich version of the vector-valued Shepard operators that have been investigated in our recent study [8]. We should note that the classical Shepard operators, which were first introduced by D. Shepard [9] in 1968, are quite effective not only in classical approximation theory (see [10–15]) but also in some applied research (see [16–18]).

Now, we first recall some notations and definitions about the vector-valued Shepard operators examined in [8].

Let $m, n \in \mathbb{N}$, $\mathbf{K} = [0,1]^m = [0,1] \times \cdots \times [0,1]$, and define the following set:

$$\Omega_n := \{\mathbf{k} = (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m : k_i \in \{0, 1, \ldots, n\},\ i = 1, 2, \ldots, m\}.$$

Then, consider the following sample points of $\mathbf{K}$:

$$\mathbf{x}_{\mathbf{k},n} = \left(\frac{k_1}{n}, \frac{k_2}{n}, \ldots, \frac{k_m}{n}\right) \text{ with } \mathbf{k} \in \Omega_n.$$

Let $\mathbf{f} = (f_1, f_2, \ldots, f_d)$ $(d \in \mathbb{N})$ be a vector-valued function defined on $\mathbf{K}$, where each component $f_i : \mathbf{K} \to \mathbb{R}$ $(i = 1, 2, \ldots, d)$. Then, for $\lambda > 0$, the vector-valued Shepard operators are defined in [8] as follows:

$$\mathbb{S}_{n,\lambda}(\mathbf{f};\mathbf{x}) = \frac{\sum_{\mathbf{k}\in\Omega_n} |\mathbf{x} - \mathbf{x}_{\mathbf{k},n}|_m^{-\lambda} \mathbf{f}(\mathbf{x}_{\mathbf{k},n})}{\sum_{\mathbf{k}\in\Omega_n} |\mathbf{x} - \mathbf{x}_{\mathbf{k},n}|_m^{-\lambda}}, \tag{1}$$

where $|\cdot|_m$ represents the classical Euclidean distance on $\mathbf{K}$. Note that the symbol $\sum_{\mathbf{k}\in\Omega_n}$ denotes the multi-index summation. We denote the space of all continuous vector-valued functions from $\mathbf{K}$ into $\mathbb{R}^d$ by $C(\mathbf{K}, \mathbb{R}^d)$. Then, in [8], we proved the following approximation result.

Theorem 1. (see Theorem 1 in [8]). *For every $\mathbf{f} \in C(\mathbf{K}, \mathbb{R}^d)$ and $\lambda \geq m + 1$, we have $\mathbb{S}_{n,\lambda}(\mathbf{f}) \rightrightarrows \mathbf{f}$ on $\mathbf{K}$, where the symbol $\rightrightarrows$ denotes the uniform convergence.*

This paper is organized as follows. In the second section, we first construct the Kantorovich version of the vector-valued Shepard operators defined by (1) and give the statements of our main theorems, including L_p-approximation, which improves Theorem 1. In the third section, we prove the theorems by using some auxiliary results. In the final section, we display some applications verifying our results and investigate the effects of nonnegative regular matrix summability methods for L_p-approximation.

2. Construction of the Operators and Main Theorems

For a given vector-valued function $\mathbf{f} = (f_1, f_2, \ldots, f_d)$, assume that each component function $f_i : \mathbf{K} \to \mathbb{R}$ $(i = 1, 2, \ldots, d)$ belongs to the space $L_p(\mathbf{K})$ $(p \geq 1)$. Then, we denote the space of all such vector-valued functions by $L_p\left(\mathbf{K}, \mathbb{R}^d\right)$. Then, we consider the following Kantorovich version of the operators (1):

$$\mathbb{L}_{n,\lambda}(\mathbf{f};\mathbf{x}) = (n+1)^m \sum_{\mathbf{k}\in\Omega_n} s_{\mathbf{k},n}(\lambda, \mathbf{x}) \int_{\mathbf{R}_{\mathbf{k},n}} \mathbf{f}(\mathbf{y}) d\mathbf{y}, \tag{2}$$

where $\mathbf{x} \in \mathbf{K}$, $n, m, d \in \mathbb{N}$, $\lambda > 0$, $\mathbf{f} = (f_1, f_2, \ldots, f_d) \in L_p\left(\mathbf{K}, \mathbb{R}^d\right)$ and

$$s_{\mathbf{k},n}(\lambda, \mathbf{x}) = \frac{|\mathbf{x} - \mathbf{x}_{\mathbf{k},n}|_m^{-\lambda}}{\sum_{\mathbf{k}\in\Omega_n} |\mathbf{x} - \mathbf{x}_{\mathbf{k},n}|_m^{-\lambda}} \text{ for } \mathbf{x} \neq \mathbf{x}_{\mathbf{j},n} \ (\mathbf{j} \in \Omega_n), \tag{3}$$

and $s_{\mathbf{k},n}(\lambda, \mathbf{x}_{\mathbf{j},n}) = \delta_{\mathbf{k},\mathbf{j}}$ with $\delta_{\mathbf{k},\mathbf{j}}$ being the Kronecker delta. The set $\mathbf{R}_{\mathbf{k},n}$ in (2) denotes the m-dimensional rectangle

$$\mathbf{R}_{\mathbf{k},n} := \left[\frac{k_1}{n+1}, \frac{k_1+1}{n+1}\right] \times \cdots \times \left[\frac{k_m}{n+1}, \frac{k_m+1}{n+1}\right]$$

and the multiple integral in (2) is actually a Bochner-type integral representation (see, for instance, [19]) and reads as follows (with respect to the components of $\mathbf{f}$) :

$$\int_{\mathbf{R}_{\mathbf{k},n}} \mathbf{f}(\mathbf{y}) d\mathbf{y} = \left(\int_{\mathbf{R}_{\mathbf{k},n}} f_1(\mathbf{y}) d\mathbf{y}, \ldots, \int_{\mathbf{R}_{\mathbf{k},n}} f_d(\mathbf{y}) d\mathbf{y} \right).$$

Then, it is easy to check that $\mathbb{L}_{n,\lambda}(\mathbf{f})$ may be written as

$$\mathbb{L}_{n,\lambda}(\mathbf{f};\mathbf{x}) = \left(\tilde{\mathbb{L}}_{n,\lambda}(f_1;\mathbf{x}), \tilde{\mathbb{L}}_{n,\lambda}(f_2;\mathbf{x}), \ldots, \tilde{\mathbb{L}}_{n,\lambda}(f_d;\mathbf{x})\right),$$

where $\tilde{\mathbb{L}}_{n,\lambda}$ is given by

$$\tilde{\mathbb{L}}_{n,\lambda}(g;\mathbf{x}) := (n+1)^m \sum_{\mathbf{k}\in\Omega_n} s_{\mathbf{k},n}(\lambda,\mathbf{x}) \int_{\mathbf{R}_{\mathbf{k},n}} g(\mathbf{y})d\mathbf{y} \tag{4}$$

for real-valued functions g defined on $\mathbf{K}$. We say that $\tilde{\mathbb{L}}_{n,\lambda}$ is the companion operator of $\mathbb{L}_{n,\lambda}$. In this case, $\tilde{\mathbb{L}}_{n,\lambda}(g;\mathbf{x})$ given by (4) becomes real-valued.

Here is our main approximation result.

Theorem 2. *For every $\mathbf{f} \in L_p\left(\mathbf{K}, \mathbb{R}^d\right)$ $(p \geq 1)$ and $\lambda \geq m+1$, we have*

$$\mathbb{L}_{n,\lambda}(\mathbf{f}) \to \mathbf{f} \ \textit{in} \ L_p\left(\mathbf{K}, \mathbb{R}^d\right) \ \textit{as} \ n \to \infty. \tag{5}$$

We should note that by the convergence in (5), we mean componentwise convergence in the space $L_p(\mathbf{K})$; that is, for each $i = 1, 2, \ldots, d$,

$$\lim_{n\to\infty} \left\| \tilde{\mathbb{L}}_{n,\lambda}(f_i) - f_i \right\|_p = 0$$

holds, where the symbol $\|\cdot\|_p$ denotes the usual L_p-norm on $\mathbf{K}$ given by

$$\|g\|_p = \left(\int_{\mathbf{K}} |g(\mathbf{y})|^p d\mathbf{y} \right)^{1/p}, \ \ p \geq 1$$

for a real-valued function $g \in L_p(\mathbf{K})$.

To prove Theorem 2, we should first show that (5) is valid for all $\mathbf{f} \in C(\mathbf{K}, \mathbb{R}^d)$. That is, we also need the next result.

Theorem 3. *For every $\mathbf{f} \in C(\mathbf{K}, \mathbb{R}^d)$ and $\lambda \geq m+1$, the convergence in (5) holds.*

3. Auxiliary Results and Proofs of the Main Theorems

To prove Theorems 2 and 3, we need the following lemmas.

Lemma 1. (see [8]). *Let $n, m \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{K}$ with $\mathbf{x} \neq \mathbf{x}_{\mathbf{k},n}$ for $\mathbf{k} \in \Omega_n$. Then, for every $\lambda > 0$,*

$$\left(\sum_{\mathbf{k}\in\Omega_n} |\mathbf{x} - \mathbf{x}_{\mathbf{k},m}|_m^{-\lambda} \right)^{-1} = O\left(n^{-\lambda}\right)$$

holds.

For the function $s_{\mathbf{k},n}(\lambda,\mathbf{x})$ given by (3), we get the next result.

Lemma 2. *For every $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbf{K}$ and $\lambda > 1$,*

$$s_{\mathbf{k},n}(\lambda,\mathbf{x}) \leq C \left\{ \sum_{i=1}^{m} |[(n+1)x_i] - k_i| + 1)^2 \right\}^{-\lambda/2} \tag{6}$$

holds for $\mathbf{k} \in \Omega_n$, where C is a positive constant depending at most on λ, m, and $[\alpha]$ is the greatest integer not exceeding α.

Proof. First, assume that $\mathbf{x} = \mathbf{x}_{\mathbf{j},n}$ $(\mathbf{j} \in \Omega_n)$. Since

$$s_{\mathbf{k},n}(\lambda, \mathbf{x}_{\mathbf{j},n}) = \delta_{\mathbf{k},\mathbf{j}} = \begin{cases} 1, & \text{if } \mathbf{k} = \mathbf{j} \\ 0, & \text{if } \mathbf{k} \neq \mathbf{j}, \end{cases}$$

the proof follows immediately. Assume now that $\mathbf{x} \neq \mathbf{x}_{\mathbf{j},n}$ $(\mathbf{j} \in \Omega_n)$. Let $[(n+1)x_i] = N_i$ for each $i = 1, 2, \ldots, m$. Then, we observe that

$$\frac{N_i}{n+1} \leq x_i < \frac{N_i + 1}{n+1} \text{ for } i = 1, 2, \ldots, m.$$

For each $i = 1, 2, \ldots, m$, we have the following five possible cases:

$$\begin{cases} k_i < N_i - 1 \\ \text{or} \\ k = N_i - 1,\ N_i,\ N_i + 1 \\ \text{or} \\ k_i > N_i + 1 \end{cases}$$

Therefore, we have a total of 5^m possible cases. After some simple computations, it is possible to check that (6) is valid for all possible cases. Now we show some of them. For example, let $k_i < N_i - 1$ for all $i = 1, 2, \ldots, m$. Lemma 1 implies that there exists a positive constant C_1 such that

$$s_{\mathbf{k},n}(\lambda, \mathbf{x}) \leq C_1 n^{-\lambda} |x - x_{\mathbf{k},n}|_m^{-\lambda} = C_1 \left\{ \sum_{i=1}^{m} (nx_i - k_i)^2 \right\}^{-\lambda/2}.$$

Then, we get

$$\begin{aligned} s_{\mathbf{k},n}(\lambda, \mathbf{x}) &\leq C_1 \left\{ \sum_{i=1}^{m} \left(\frac{nN_i}{n+1} - k_i \right)^2 \right\}^{-\lambda/2} \\ &= C_1 \left\{ \sum_{i=1}^{m} \left(\frac{n(N_i - k_i) - k_i}{n+1} \right)^2 \right\}^{-\lambda/2} \\ &\leq C_1 \left\{ \sum_{i=1}^{m} (N_i - k_i - 1)^2 \right\}^{-\lambda/2} \\ &\leq C_1 \left\{ \sum_{i=1}^{m} \left(\frac{N_i - k_i + 1}{4} \right)^2 \right\}^{-\lambda/2} \\ &= C \left\{ \sum_{i=1}^{m} (|[(n+1)x_i] - k_i| + 1)^2 \right\}^{-\lambda/2}, \end{aligned}$$

where $C := 4^\lambda C_1$. Now, for some $m_0 \in \{1, 2, \ldots, m-1\}$, if $k_i < N_i - 1$ for $i = 1, 2, \ldots, m_0$ and $k_i > N_i + 1$ for $i = m_0 + 1, \ldots, m$, then using the same constants C_1 and C, we see that

$$
\begin{aligned}
s_{\mathbf{k},n}(\lambda,\mathbf{x}) &\leq C_1\left\{\sum_{i=1}^{m}(nx_i-k_i)^2\right\}^{-\lambda/2} \\
&\leq C_1\left\{\sum_{i=1}^{m_0}\left(\frac{nN_i}{n+1}-k_i\right)^2+\sum_{i=m_0+1}^{m}\left(k_i-\frac{n(N_i+1)}{n+1}\right)^2\right\}^{-\lambda/2} \\
&= C_1\left\{\sum_{i=1}^{m_0}\left(\frac{n(N_i-k_i)-k_i}{n+1}\right)^2+\sum_{i=m_0+1}^{m}\left(\frac{n(k_i-N_i-1)+k_i}{n+1}\right)^2\right\}^{-\lambda/2} \\
&\leq C_1\left\{\sum_{i=1}^{m_0}(N_i-k_i-1)^2+\sum_{i=m_0+1}^{m}(k_i-N_i-1)^2\right\}^{-\lambda/2} \\
&\leq C_1\left\{\sum_{i=1}^{m_0}\left(\frac{N_i-k_i+1}{4}\right)^2+\left(\frac{k_i-N_i+1}{4}\right)^2\right\}^{-\lambda/2} \\
&= C\left\{\sum_{i=1}^{m}(|[(n+1)x_i]-k_i|+1)^2\right\}^{-\lambda/2}.
\end{aligned}
$$

Now let $k_i = N_i - 1$ for all $i = 1, 2, \ldots, m$. Then we observe that

$$
\begin{aligned}
s_{\mathbf{k},n}(\lambda,\mathbf{x}) &\leq 1 \\
&= (4m)^{\lambda/2}\left\{\sum_{i=1}^{m}(1+1)^2\right\}^{-\lambda/2} \\
&= (4m)^{\lambda/2}\left\{\sum_{i=1}^{m}(|N_i-(N_i-1)|+1)^2\right\}^{-\lambda/2} \\
&= (4m)^{\lambda/2}\left\{\sum_{i=1}^{m}(|[(n+1)x_i]-k_i|+1)^2\right\}^{-\lambda/2}.
\end{aligned}
$$

Also, for a given $m_0 \in \{1, 2, \ldots, m-1\}$, if $k_i = N_i$ for $i = 1, 2, \ldots, m_0$ and $k_i > N_i + 1$ for $i = m_0 + 1, \ldots, n$, we may then write that

$$
\begin{aligned}
s_{\mathbf{k},n}(\lambda,\mathbf{x}) &\leq C_1\left\{\sum_{i=m_0+1}^{m}(nx_i-k_i)^2\right\}^{-\lambda/2} \\
&\leq C_1\left\{\sum_{i=m_0+1}^{m}\left(\frac{nN_i}{n+1}-k_i\right)^2\right\}^{-\lambda/2} \\
&\leq C_1\left\{\sum_{i=m_0+1}^{m}(|[(n+1)x_i]-k_i|+1)^2\right\}^{-\lambda/2} \\
&\leq \left(\frac{m}{m-m_0}\right)^{\lambda/2}C_1\left\{m_0+\sum_{i=m_0+1}^{m}(|[(n+1)x_i]-k_i|+1)^2\right\}^{-\lambda/2} \\
&= \left(\frac{m}{m-m_0}\right)^{\lambda/2}C_1\left\{\sum_{i=1}^{m_0}(|N_i-N_i|+1)^2+\sum_{i=m_0+1}^{m}(|[(n+1)x_i]-k_i|+1)^2\right\}^{-\lambda/2} \\
&= C\left\{\sum_{i=1}^{m}(|[(n+1)x_i]-k_i|+1)^2\right\}^{-\lambda/2},
\end{aligned}
$$

where $C = \left(\frac{m}{m-m_0}\right)^{\lambda/2}C_1$. By making similar calculations, it can be shown that (6) holds true in all other cases. □

Now for each fixed $\mathbf{x} \in \mathbf{K}$, define the function $\varphi_{\mathbf{x}}$ on $\mathbf{K}$ by

$$\varphi_{\mathbf{x}}(\mathbf{y}) := |\mathbf{y} - \mathbf{x}|_m.$$

Then, we get the next lemma.

Lemma 3. *For any $\mathbf{x} \in \mathbf{K}$, we have*

$$\tilde{\mathbb{L}}_{n,\lambda}(\varphi_{\mathbf{x}};\mathbf{x}) = \begin{cases} O(n^{-1}), & \text{if } \lambda > m+1 \\ O(n^{-1}\log n), & \text{if } \lambda = m+1. \end{cases}$$

Proof. For a given $n \in \mathbb{N}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbf{K}$, there exists $\mathbf{u} = (u_1, u_2, \ldots, u_m) \in \Omega_n$ such that $x_i \in \left[\frac{u_i}{n+1}, \frac{u_i+1}{n+1}\right]$ for $i = 1, 2, \ldots, m$. Hence, Lemma 2 implies that

$$s_{\mathbf{k},n}(\lambda,\mathbf{x}) \le C\left\{\sum_{i=1}^{m}(|u_i - k_i| + 1)^2\right\}^{-\lambda/2}.$$

Then, we get

$$\begin{aligned} \tilde{\mathbb{L}}_{n,\lambda}(\varphi_{\mathbf{x}};\mathbf{x}) &= (n+1)^m \sum_{\mathbf{k}\in\Omega_n} s_{\mathbf{k},n}(\lambda,\mathbf{x}) \int_{\mathbf{R}_{\mathbf{k},n}} |\mathbf{y}-\mathbf{x}|_m d\mathbf{y} \\ &\le (n+1)^m \sum_{\mathbf{k}\in\Omega_n} \frac{s_{\mathbf{k},n}(\lambda,\mathbf{x})}{(n+1)^{m+1}} \left\{\sum_{i=1}^{m}(|u_i - k_i| + 1)^2\right\}^{1/2} \\ &\le \frac{C}{n+1} \sum_{\mathbf{k}\in\Omega_n} \left\{\sum_{i=1}^{m}(|u_i - k_i| + 1)^2\right\}^{(1-\lambda)/2} \\ &\le \frac{C}{n+1} \sum_{k_1,k_2,\ldots,k_m=1}^{n} \frac{1}{\left(k_1^2 + k_2^2 + \cdots + k_m{}^2\right)^{(\lambda-1)/2}}. \end{aligned}$$

We know from Lemma 2.2 in [8] and its conclusion that

$$\sum_{k_1,k_2,\ldots,k_m=1}^{n} \frac{1}{\left(k_1^2 + k_2^2 + \cdots + k_m{}^2\right)^{(\lambda-1)/2}} = \begin{cases} O(1), & \text{if } \lambda > m+1 \\ O(\log n), & \text{if } \lambda = m+1. \end{cases}$$

Therefore, by combining the above results, the proof is completed. □

With the help of the above lemmas, we first prove Theorem 3.

Proof of Theorem 3. Let $\mathbf{f} = (f_1, f_2, \ldots, f_d) \in C(\mathbf{K}, \mathbb{R}^d)$ and $\lambda \ge m+1$. By the uniform continuity of each component f_i $(i = 1, 2, \ldots, d)$ on $\mathbf{K}$, for every $\varepsilon > 0$, there exists a $\delta_i > 0$ such that

$$|f_i(\mathbf{y}) - f_i(\mathbf{x})| < \varepsilon$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{K}$ satisfying $|\mathbf{y} - \mathbf{x}|_m < \delta_i$. Then, it follows from (4) that for each $i = 1, 2, \ldots, d$,

$$\begin{aligned} \left|\tilde{\mathbb{L}}_{n,\lambda}(f_i;\mathbf{x}) - f_i(\mathbf{x})\right| &\le (n+1)^m \sum_{\mathbf{k}\in\Omega_n} s_{\mathbf{k},n}(\lambda,\mathbf{x}) \int_{\mathbf{K}} |f_i(\mathbf{y}) - f_i(\mathbf{x})| d\mathbf{y} \\ &\le (n+1)^m \sum_{\mathbf{k}\in\Omega_n} s_{\mathbf{k},n}(\lambda,\mathbf{x}) \int_{\mathbf{K}} \left(\varepsilon + \frac{2M}{\delta_i}|\mathbf{y}-\mathbf{x}|_m\right) d\mathbf{y} \\ &= \varepsilon + \frac{2M_i}{\delta_i} \tilde{\mathbb{L}}_{n,\lambda}(\varphi_{\mathbf{x}};\mathbf{x}). \end{aligned}$$

Lemma 3 implies that for each $i = 1, 2, \ldots, d$,

$$\tilde{\mathbb{L}}_{n,\lambda}(f_i) \rightrightarrows f_i \text{ on } \mathbf{K}$$

holds for $\lambda \geq m+1$. Since the uniform convergence on $\mathbf{K}$ implies L_p-convergence, we obtain for each $i = 1, 2, \ldots, m$ that

$$\lim_{n\to\infty} \left\| \tilde{\mathbb{L}}_{n,\lambda}(f_i) - f_i \right\|_p = 0$$

holds for $\lambda \geq m+1$, which completes the proof. □

For the proof of Theorem 2, we also need the next lemma.

Lemma 4. *Let $\lambda \geq m+1$ and $p \geq 1$. Then, the sequence of companion operators $\{\tilde{\mathbb{L}}_{n,\lambda}\}$ given by (4) is uniformly bounded from $L_p(\mathbf{K})$ into itself, i.e., for every $g \in L_p(\mathbf{K})$,*

$$\left\| \tilde{\mathbb{L}}_{n,\lambda}(g) \right\|_p \leq B \|g\|_p$$

holds for some absolute constant B.

Proof. Lemma 2 immediately gives that for every $\mathbf{k} \in \Omega_n$,

$$\begin{aligned} \int_{\mathbf{K}} s_{\mathbf{k},n}(\lambda, \mathbf{x}) d\mathbf{x} &= \sum_{\mathbf{u}\in\Omega_n} \int_{\mathbf{R}_{\mathbf{u},n}} s_{\mathbf{k},n}(\lambda, \mathbf{x}) d\mathbf{x} \\ &\leq C \sum_{\mathbf{u}\in\Omega_n} \int_{\mathbf{R}_{\mathbf{u},n}} \left\{ \sum_{i=1}^{m} (|u_i - k_i| + 1)^2 \right\}^{-\lambda/2} d\mathbf{x} \\ &= \frac{C}{(n+1)^m} \sum_{\mathbf{u}\in\Omega_n} \left\{ \sum_{i=1}^{m} (|u_i - k_i| + 1)^2 \right\}^{-\lambda/2} \\ &\leq \frac{C}{(n+1)^m} \sum_{k_1,k_2,\ldots,k_m=1}^{n} \frac{1}{(k_1^2 + k_2^2 + \cdots + k_m{}^2)^{\lambda/2}} \\ &= O\left(\frac{1}{(n+1)^m} \right) \end{aligned}$$

holds for $\lambda \geq m+1$. If $g \in L_1(\mathbf{K})$, then we obtain that

$$\begin{aligned} \left\| \tilde{\mathbb{L}}_{n,\lambda}(g) \right\|_1 &= \int_{\mathbf{K}} \left| \tilde{\mathbb{L}}_{n,\lambda}(g; \mathbf{x}) \right| d\mathbf{x} \\ &\leq (n+1)^m \sum_{\mathbf{k}\in\Omega_n} \left(\int_{\mathbf{R}_{\mathbf{k},n}} |g(\mathbf{y})| d\mathbf{y} \right) \int_{\mathbf{K}} s_{\mathbf{k},n}(\lambda, \mathbf{x}) d\mathbf{x} \\ &\leq C \sum_{\mathbf{k}\in\Omega_n} \left(\int_{\mathbf{R}_{\mathbf{k},n}} |g(\mathbf{y})| d\mathbf{y} \right), \end{aligned}$$

which yields

$$\left\| \tilde{\mathbb{L}}_{n,\lambda}(g) \right\|_1 \leq C \|g\|_1 \text{ for } \lambda \geq m+1. \tag{7}$$

On the other hand, if $g \in C(\mathbf{K})$, then one can easily check that

$$\left\| \tilde{\mathbb{L}}_{n,\lambda}(g) \right\| \leq \|f\|, \tag{8}$$

where the symbol $\|\cdot\|$ denotes the usual supremum norm on $\mathbf{K}$. Therefore, considering (7) and (8), the Riesz–Thorin theorem [20] (see also [15]) implies that for some absolute constant $B > 0$,

$$\left\| \tilde{\mathbb{L}}_{n,\lambda}(g) \right\|_p \leq B \|g\|_p$$

is satisfied for every $g \in L_p(\mathbf{K})$ $(p \geq 1)$ and $\lambda \geq m+1$. □

Then, we are ready to give the proof of our main theorem.

Proof of Theorem 2. Let $\mathbf{f} \in L^p(\mathbf{K}, \mathbb{R}^d)$ $(p \geq 1)$. Then for each component $f_i \in L_p(\mathbf{K})$, $i = 1, 2, \ldots, d$, there exists a real-valued continuous function g_i on $\mathbf{K}$ such that

$$\begin{aligned} \left\|\tilde{\mathbb{L}}_{n,\lambda}(f_i) - f_i\right\|_p \quad &\leq \quad \left\|\tilde{\mathbb{L}}_{n,\lambda}(f_i - g_i)\right\|_p + \left\|\tilde{\mathbb{L}}_{n,\lambda}(g_i) - g_i\right\|_p \\ &\quad + \|f_i - g_i\|_p. \end{aligned}$$

Then, we may write from Lemma 4 that, for every $\lambda \geq m+1$,

$$\left\|\tilde{\mathbb{L}}_{n,\lambda}(f_i) - f_i\right\|_p \leq C\|f_i - g_i\|_p + \left\|\tilde{\mathbb{L}}_{n,\lambda}(g_i) - g_i\right\|_p \tag{9}$$

holds for some $C > 0$. From Theorem 3, we get

$$\lim_{n\to\infty} \left\|\tilde{\mathbb{L}}_{n,\lambda}(g_i) - g_i\right\|_p = 0. \tag{10}$$

Now, since the space of all real-valued and continuous functions on $\mathbf{K}$ is dense in the space $L_p(\mathbf{K})$, the proof is completed from (9) and (10). □

4. Illustrations and Concluding Remarks

We first give applications of Theorems 2 and 3 on the set $K = [0,1]^m$. Later, we modify vector-valued Shepard operators in order to show the effects of regular summability methods in the approximation.

Example 1. *Take $d = 3$ and $m = 2$. Define the function $\mathbf{f}$ on $\mathbf{K} = [0,1]^2$ by*

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})),$$

where for $\mathbf{x} = (x_1, x_2) \in \mathbf{K}$, the component functions are given, respectively, by

$$\begin{aligned} f_1(\mathbf{x}) &= x_1, \\ f_2(\mathbf{x}) &= x_2, \\ f_3(\mathbf{x}) &= [x_1 + x_2]. \end{aligned} \tag{11}$$

Then, we obtain from Theorem 2 that for every $\lambda \geq 3$ and $p \geq 1$,

$$\mathbb{L}_{n,\lambda}(\mathbf{f}) \to \mathbf{f} \ \ in \ \ L_p\left([0,1]^2, \mathbb{R}^3\right) \ \ as \ \ n \to \infty.$$

If the function $\mathbf{f}$ is considered to be a three-dimensional surface parametrized by x_1 and x_2, one can produce its three-dimensional parametric plots with the help of the Mathematica program. Similarly, we can also produce the corresponding approximations by vector-valued Shepard operators. Such parametric plots are shown in Figure 1 for the values $n = 5, 12, 20$ and $\lambda = 6$. Observe that since $\mathbf{f}$ is not continuous on $\mathbf{K}$, Theorem 1 is not valid for the function $\mathbf{f}$ given by (11). Hence, this example explains why we also need the Kantorovich version of vector-valued Shepard operators.

Example 2. *Take $d = 3$ and $m = 1$. Now define the function $\mathbf{h}$ on the set $\mathbf{K} = [0,1]$ by*

$$\mathbf{h}(x) = (\sin(20x), \cos(20x), 2x). \tag{12}$$

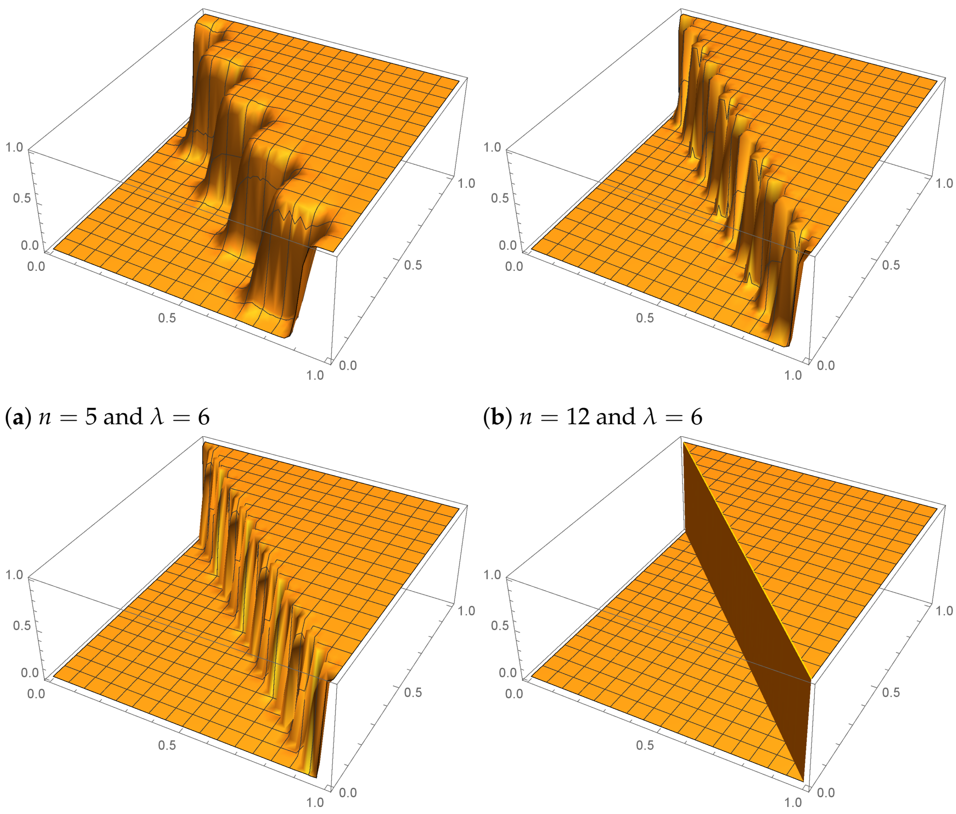
Then this function parametrized by x gives a helix curve. Since $\mathbf{h} \in C\left(\mathbf{K}, \mathbb{R}^3\right)$, we obtain from Theorems 1 and 3 that for every $\lambda \geq 2$,

$$\mathbb{L}_{n,\lambda}(\mathbf{h}) \rightrightarrows \mathbf{h} \ \ on \ \ [0,1]$$

and

$$\mathbb{L}_{n,\lambda}(\mathbf{h}) \to \mathbf{h} \ \ in \ \ L_p\Big([0,1], \mathbb{R}^3\Big).$$

This approximation is indicated in Figure 2 for the values $n = 15, 22, 30$ *and* $\lambda = 4$.

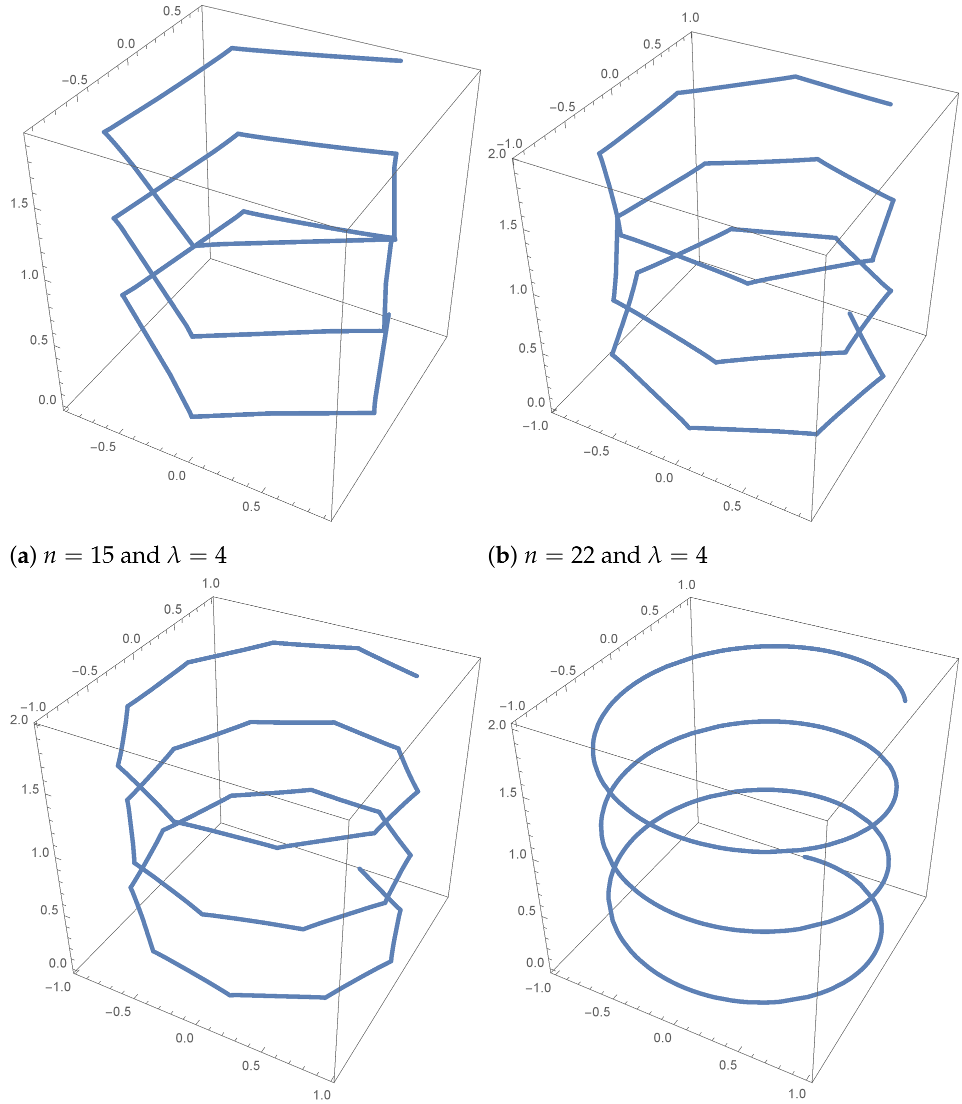


(**a**) $n = 5$ and $\lambda = 6$ (**b**) $n = 12$ and $\lambda = 6$

(**c**) $n = 20$ and $\lambda = 6$ (**d**) $\mathbf{f}$

Figure 1. Parametric plots of $\mathbb{L}_{n,\lambda}(\mathbf{f})$ for the values $n = 5, 12, 20$ and $\lambda = 6$, where $\mathbf{f}$ is given by (11).

Finally, we discuss the regular summability methods on the L_p-approximation. Before giving our final application, we recall some concepts from summability theory. For a given infinite matrix $A := [a_{jn}]$ $(j, n \in \mathbb{N})$ and a sequence $x := (x_n)$, the A-transformed sequence of (x_n) is defined by $Ax := \big((Ax)_j\big) = \sum_{n=1}^{\infty} a_{jn} x_n$ provided that the series is convergent for every $j \in \mathbb{N}$. Also, $A = [a_{jn}]$ is called regular if $\lim Ax = L$ whenever $\lim x = L$ (see [21]). $A = [a_{jn}]$ is nonnegative if $a_{jn} \geq 0$ for all $j, n \in \mathbb{N}$. Now let $A = [a_{jn}]$ be a nonnegative regular summability matrix. Then, we say that a sequence (x_n) is A-summable (or A-convergent) to a number L if $\lim_{n\to\infty}(Ax)_j = L$. It is also possible to give the same definition for a sequence of functions in the space $L_p\Big(\mathbf{K}, \mathbb{R}^d\Big)$ $(p \geq 1)$. Let $(\mathbf{f}_n)$ be a sequence of vector-valued functions in $L_p\Big(\mathbf{K}, \mathbb{R}^d\Big)$, and let $A = [a_{jn}]$ be a nonnegative regular summability method such that $\sum_{n=1}^{\infty} a_{jn} \mathbf{f}_n \in L_p\Big(\mathbf{K}, \mathbb{R}^d\Big)$ for every $j \in \mathbb{N}$. Then, we say that (f_n) is A-summable to a function $\mathbf{f}$ in $L_p\Big(\mathbf{K}, \mathbb{R}^d\Big)$ if $\sum_{n=1}^{\infty} a_{jn} \mathbf{f}_n \to \mathbf{f}$ in $L_p\Big(\mathbf{K}, \mathbb{R}^d\Big)$ as $j \to \infty$. As stated before, here we mean the componentwise L_p-convergence on $\mathbf{K}$.



(**a**) $n = 15$ and $\lambda = 4$ (**b**) $n = 22$ and $\lambda = 4$

(**c**) $n = 30$ and $\lambda = 4$ (**d**) $\mathbf{h}$

Figure 2. Parametric plots of $\mathbb{L}_{n,\lambda}(\mathbf{h})$ for the values $n = 15, 22, 30$ and $\lambda = 4$, where $\mathbf{h}$ is given by (12).

We should note that the use of regular summability methods in the approximation theory enables us to get more powerful results than the classical ones. We will now consider an application in this direction.

Example 3. *In this application, we modify the vector-valued Kantorovich–Shepard operators in (2) as follows:*

$$\mathbb{L}^*_{n,\lambda}(\mathbf{f};\mathbf{x}) := \begin{cases} \mathbf{1} + \mathbf{f}(\mathbf{x}), & \text{if } n = k^2 \ (k \in \mathbb{N}) \\ \mathbb{L}_{n,\lambda}(\mathbf{f};\mathbf{x}) & \text{otherwise,} \end{cases} \tag{13}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$. *Since* $\mathbf{1} + \mathbf{f} \neq \mathbf{f}$, *we cannot get an* L_p*-approximation to* $\mathbf{f}$ *by means of the operators* $\mathbb{L}^*_{n,\lambda}(\mathbf{f})$ *given by (13); that is, for every* $\lambda > 0$ *and* $i = 1, 2, \ldots, d$,

$$\mathbb{L}^*_{n,\lambda}(\mathbf{f}) \nrightarrow \mathbf{f} \ \text{ in } \ L_p\left(\mathbf{K}, \mathbb{R}^d\right) \ \text{ as } n \to \infty.$$

Now to overcome the loss of convergence, we consider the well-known Cesàro summability method $C_1 := [c_{jn}]$ (see [21] for details) given by

$$c_{jn} = \begin{cases} \frac{1}{j}, & \text{if } n = 1, 2, \ldots, j \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{f} \in L_p\left(K, \mathbb{R}^d\right)$ $(p \geq 1)$ and $\lambda \geq m+1$ be given. Then, we observe that the arithmetic mean of $\mathbb{L}^*_{n,\lambda}(\mathbf{f})$ is L_p-convergent to $\mathbf{f}$ in $L_p\left(\mathbf{K}, \mathbb{R}^d\right)$. To see that considering the companion operator $\tilde{L}^*_{n,\lambda}$ of (13), it is enough to show that for each $i = 1, 2, \ldots, d$, the sequence $\left(\tilde{\mathbb{L}}^*_{n,\lambda}(f_i)\right)$, $i = 1, 2, \ldots, d$, is C_1-summable (with respect to the L_p-norm on $\mathbf{K}$) to the function f_i. Indeed, by using (13) we may write that

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} c_{jn} \tilde{\mathbb{L}}^*_{n,\lambda}(f_i) - f_i \right\|_p &= \left\| \frac{1}{j} \sum_{n=1}^{j} \tilde{\mathbb{L}}^*_{n,\lambda}(f_i) - f_i \right\|_p \\ &\leq \frac{1}{j} \sum_{n=1}^{j} \left\| \tilde{L}^*_{n,\lambda}(f_i) - f_i \right\|_p \\ &= \frac{1}{j} \sum_{n=1\ (n=k^2)}^{j} \left\| \tilde{L}^*_{n,\lambda}(f_i) - f_i \right\|_p + \frac{1}{j} \sum_{n=1\ (n \neq k^2)}^{j} \left\| \tilde{\mathbb{L}}^*_{n,\lambda}(f_i) - f_i \right\|_p \\ &\leq \frac{1}{\sqrt{j}} + \frac{1}{j} \sum_{n=1}^{j} \left\| \tilde{\mathbb{L}}_{n,\lambda}(f_i) - f_i \right\|_p, \end{aligned}$$

where $\tilde{\mathbb{L}}_{n,\lambda}$ is the classical companion operator given by (4). Now, by taking the limit as $j \to \infty$ on both sides of the last inequality, we obtain from Theorem 2 and the regularity of the Cesàro method that for each $i = 1, 2, \ldots, d$,

$$\lim_{j \to \infty} \left\| \sum_{n=1}^{\infty} c_{jn} \tilde{\mathbb{L}}^*_{n,\lambda}(f_i) - f_i \right\|_p = 0,$$

holds, which means

$$\frac{\mathbb{L}^*_{1,\lambda}(\mathbf{f}) + \mathbb{L}^*_{2,\lambda}(\mathbf{f}) + \cdots + \mathbb{L}^*_{j,\lambda}(\mathbf{f})}{j} \to \mathbf{f} \ \text{ in } \ L_p\left(\mathbf{K}, \mathbb{R}^d\right) \ \text{ as } j \to \infty.$$

In other words, the sequence $\left(\mathbb{L}^*_{n,\lambda}(\mathbf{f})\right)$ is C_1-summable to $\mathbf{f}$ in $L_p\left(\mathbf{K}, \mathbb{R}^d\right)$.

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