Chaos

Asset price dynamics for a two-asset market system

Cite as: Chaos **29**, 023114 (2019); https://doi.org/10.1063/1.5046925
Submitted: 03 July 2018 . Accepted: 14 January 2019 . Published Online: 11 February 2019

H. Bulut, H. Merdan, and D. Swigon

Asset price dynamics for a two-asset market system

Cite as: Chaos **29**, 023114 (2019); doi: 10.1063/1.5046925
Submitted: 3 July 2018 · Accepted: 14 January 2019 ·
Published Online: 11 February 2019

View Online Export Citation CrossMark

H. Bulut,[1,a)] H. Merdan,[1,b)] and D. Swigon[2,c)]

AFFILIATIONS

[1] Department of Mathematics, TOBB University of Economics and Technology, 06560 Ankara, Turkey
[2] Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA

[a)] E-mail: hcakar@etu.edu.tr
[b)] E-mail: merdan@etu.edu.tr
[c)] E-mail: swigon@pitt.edu

ABSTRACT

We present a mathematical model for a market involving two stocks which are traded within a single homogeneous group of investors who have similar motivations and strategies for trading. It is assumed that the market consists of a fixed amount of cash and stocks (additions in time are not allowed, so the system is closed) and that the trading group is affected by trend and valuation motivations while selling or buying each asset, but follows a strategy in which the buying of an asset depends on the other asset's price while the selling does not. By utilizing these assumptions and basic microeconomics principles, the mathematical model is obtained through a dynamical system approach. We analyze the stability of equilibrium points of the model and determine the conditions on parameters for stability. First, we prove that all equilibria are stable in the absence of a clear emphasis on a trend-based value for each stock. Second, for systems in which the group of traders attaches importance to the valuation of one stock and the trend of the other stock for trading, we establish conditions for stability and show with numerical examples that when instability occurs, it is exhibited by oscillations in the price of both stocks. Moreover, we argue the existence of periodic solutions through a Hopf bifurcation by choosing the momentum coefficient as a bifurcation parameter within this setting. Finally, we give examples and numerical simulations to support and extend the analytical results. One of the key conclusions for economics and finance is the existence of a cyclic behavior in the absence of exogenous factors according to the momentum coefficient. In particular, an equilibrium price that is stable becomes unstable as the trend based trading increases.

The instabilities of financial markets cause great harm to the economies of countries, and market analysts and policy-makers often discuss these issues. Mathematical modeling of the financial markets may improve the understanding of the dynamics of the markets and offer applicable solutions to issues in them. In this direction, stochastic models and deterministic models have been introduced. The stochastic models are derived from theoretical assumptions and largely based on the efficient market hypothesis, so they treat instabilities as rare occurrences. Moreover, these models do not offer oscillations or cyclic behavior within this setting. However, the deterministic models, which have emerged as alternative perspectives on asset price dynamics in 1990, often analyze practical issues such as the market crashes and the effect of excess cash. Using the theory of differential equations, one can study stabilities and cyclic behavior of these models. While the dynamics of a financial market consisting of a single stock has been explored mathematically, and the conditions for its stability and instability are understood, it is not known whether such instabilities can influence the markets for other stocks. The aim of this paper is to introduce a mathematical model that is capable of addressing stabilities, instabilities, and also cyclic behaviors within a financial market with multiple stocks traded simultaneously. Toward this goal, we present a deterministic model for a two-asset market system using a dynamical system approach. It is assumed that the market involves a homogeneous group of investors who have the same characteristics for trading. The model, besides basic microeconomics principles, is derived based on several key aspects (for example, the finiteness of assets and the existence of the different motivations and strategies for trading)

that are routinely examined by practitioners. Stability analyses and numerical studies imply that equilibrium prices are stable if the group of investors focuses on only the fundamental values of the assets being traded. In contrast, trading that is largely affected by momentum effects leads to instabilities in the asset prices, which are characteristics of the crises of financial markets such as the high-tech bubble of 1998-2000 in the United States. Another important result is the possibility of the existence of periodic solutions that are not permitted in classical finance theory.

I. INTRODUCTION

Local and global issues arising in financial markets affect the dynamics and stability of those markets and underline the need for developing mathematical models that are capable of suggesting solutions for such disturbances. In this direction, a multitude of stochastic and deterministic asset pricing models have been introduced. Within the stochastic settings, the models are derived based on the efficient market hypothesis combined with the following assumptions: (i) the supply and demand are based upon the value of the asset, (ii) there is a general agreement on the valuation of an asset among market participants since the information is public (so a unique price is determined by the market participants), and (iii) there is an assumption of an infinite amount of arbitrage capital that would take advantage of any discrepancies from this unique price.[1–3] In these models, the price of an asset is often determined by the following stochastic equation:

$$d\log[P(t)] = \mu dt + \sigma dW(t), \tag{1}$$

where $d\log[P(t)]$ represents the relative price change, W is the Brownian motion, μ is the expected return, and σ is the standard deviation.[4] Even though these assumptions are good idealizations for theoretical studies in classical finance, they have been criticized by some scholars.[5–15] Using the dynamical system approach, an alternative perspective was introduced to study the asset price dynamics within deterministic setting.[14,16–18] Unlike the stochastic models, these dynamical system models are derived from more realistic assumptions: (i) the value of an asset depends on not only the valuation of the asset but also other factors, including the derivative history of the price which is also called the momentum effect,[9,10,13] (ii) each investor has a different motivation and strategy for trading that eliminates the unique price argument,[7,9,11] and (iii) there is an assumption of the finiteness of assets, which ignores the arbitrage argument.[6,14,15] The deterministic models arise as a system of nonlinear differential equations which include the excess demand equation that governs the price of an asset and can be written in continuous-time form as follows:

$$\tau \frac{1}{P}\frac{dP}{dt} = \frac{D(P) - S(P)}{D(P)}, \tag{2}$$

where $D[P(t)]$ and $S[P(t)]$ are the demand and supply functions of price, $P(t)$, respectively, and τ is a proportionality constant that scales the time variable.[3,19]

Using the asset flow approach, Caginalp and his collaborators have derived dynamical systems of this type and used them to study the financial market dynamics and stability.[14–18,20–27] These early models capture the dynamics of a single asset market with a prescribed number of shares and cash supply (including additions in time) which are distributed to a homogenous group of traders randomly. The models are derived based on the assumptions of the basic conservation of cash and asset and microeconomics identities including the excess demand equation (2).[14,16–18,25] The models are also combined with the finiteness of assets, which ignores the arbitrage argument, and also different motivations and strategies in the trading that eliminate the unique price argument. By deriving a system of nonlinear differential equations, these models have been used to study a variety of issues including the forecasting of the asset pricing, some qualitative and quantitative properties of price dynamics, price patterns, over/under valuations, market bubbles, and crashes.[10,14,18,20,25,28–30] Later models, constructed under the same assumptions, have been extended to a market system with a single asset, but multiple groups of investors.[6,21,27,31] These models were used to study the dynamics of a single asset market involving a heterogeneous group of investors who all have different strategies, motivations, and also budgets. Using these multi-group models, various phenomena arising in closed end funds have been analyzed, including the price change due to a change in the number of shares,[6] and the optimality of the constant rebalanced the portfolio strategy.[31] Stability analysis, the emergence of the periodic solutions via a Hopf bifurcation, and some other bifurcation properties of these models have also been studied by several researchers.[15,21,26]

In this paper, we follow in the footsteps of these earlier models, but we focus on the price dynamics of a two-asset market. We still assume that the number of shares of each asset and the amount of cash are constant in time, but the trading strategies of each trader now depend on both stock prices, which couple their resulting dynamics. The model is an extension of the models derived for a single asset market system in Refs. 16 and 14. It is derived by assuming similar conditions including the assumption of the finiteness of the assets so unlimited arbitrage is not possible. We consider a system involving two assets traded by a homogeneous group of investors. It is assumed that there is $N^{(1)}$ shares of stock 1, $N^{(2)}$ shares of stock 2, and M units of cash in the system, which are distributed to investors randomly. These investors follow a trading strategy in which the buying of an asset depends on the other asset's price while the selling does not. With respect to this trading strategy, the investor group has preference functions for each stock that are influenced by price momentum and discount from fundamental value. Using the basic microeconomics principle together with Eq. (2), we derive a complete system of the first order non-linear differential equations. We present the equilibrium stability analysis of the model for several cases and use numerical simulations to support and extend the analytical results.

In particular, we prove that all equilibria are stable if the investor group buys and sells both assets according to only valuation. On the other hand, if the investor group utilizes the valuation for one asset and the trend effects on the trading price for the other, then the stability of the equilibrium point of the system is lost as the momentum coefficient increases. For the latter case, the existence of periodic solutions through a Hopf bifurcation is shown by choosing the momentum coefficient of one of the stocks as a bifurcation parameter. Numerical simulations support these analytical results and show that periodic solution may appear as the momentum coefficient passes through a critical value (see Fig. 8). The existence or nonexistence of limit cycles, however, depends on the details of the transition rate functions.

The paper is organized as follows. Section II presents the mathematical model. Sections III and IV consist of a stability and bifurcation analysis of the model. In Sec. IV, we also give numerical simulations to support and extend the analytical results. Section VI is devoted to results and discussions.

II. THE MODEL

We consider a market involving two stocks, namely, stock 1 and stock 2, traded within a single homogeneous group of investors, i.e., investors who share their trading strategies and preferences. It is assumed that the market involves M units of cash, $N^{(1)}$ units of stock 1, and $N^{(2)}$ units of stock 2. We assume that the trading group follows a strategy in which the buying of an asset depends on the other asset's price while the selling does not. With respect to our assumption on the trading strategy, we define transition rate functions as follows:

$$\begin{cases} k^{(1)}(t) := k^{(1)}[\zeta_1^{(1)}(t), \zeta_2^{(1)}(t), \zeta_1^{(2)}(t), \zeta_2^{(2)}(t)], \\ k^{(2)}(t) := k^{(2)}[\zeta_1^{(1)}(t), \zeta_2^{(1)}(t), \zeta_1^{(2)}(t), \zeta_2^{(2)}(t)], \\ \widetilde{k}^{(1)}(t) := \widetilde{k}^{(1)}[\zeta_1^{(1)}(t), \zeta_2^{(1)}(t)], \\ \widetilde{k}^{(2)}(t) := \widetilde{k}^{(2)}[\zeta_1^{(2)}(t), \zeta_2^{(2)}(t)], \end{cases} \tag{3}$$

where both $k^{(1)}$ and $k^{(2)}$ denote the transition rate functions from cash to stocks 1 and 2, respectively, while both $\widetilde{k}^{(1)}$ and $\widetilde{k}^{(2)}$ denote the transition rate functions from stocks 1 and 2 to cash. From another perspective, $k^{(1)}$ and $k^{(2)}$ can be defined as the probabilities that one unit of cash will be used to place an order to buy one unit of stock 1 and that of stock 2, respectively. Similarly, $\widetilde{k}^{(1)}$ and $\widetilde{k}^{(2)}$ represent the probabilities of selling of each stock, respectively, as in Refs. 6 and 14. Thus, $k^{(1)}$, $k^{(2)}$, $\widetilde{k}^{(1)}$, $\widetilde{k}^{(2)} \in [0,1]$, and $0 \leq k^{(1)} + k^{(2)} \leq 1$. The functions (3) describe how investors' decisions to buy or sell stocks depend on the quantities $\zeta_j^{(i)}$, which represent the sentiments toward each stock, where $i = 1, 2$ represents the stock number and j represents the trend-based component ($j = 1$) and the value-based component ($j = 2$) of the sentiment. Specifically, $\zeta_1^{(i)}(t)$ is the sum of all impacts of the relative price changes before time t for stock i, while $\zeta_2^{(i)}(t)$ represents investors' focus on the deviation between the asset price and its fundamental (true) value. Trading sentiments reflect the dynamics of the price of a stock and its contribution to the investors decisions on stock purchases. The sentiment functions are mathematically defined as follows:

$$\zeta_1^{(i)}(t) := q_1^{(i)} c_1^{(i)} \int_{-\infty}^{t} \frac{1}{P^{(i)}(\tau)} \frac{dP^{(i)}(\tau)}{d\tau} e^{-c_1^{(i)}(t-\tau)} d\tau, \tag{4}$$

$$\zeta_2^{(i)}(t) := q_2^{(i)} c_2^{(i)} \int_{-\infty}^{t} \frac{P_a^{(i)}(\tau) - P^{(i)}(\tau)}{P_a^{(i)}(\tau)} e^{-c_2^{(i)}(t-\tau)} d\tau, \tag{5}$$

where, for $i = 1, 2$, $c_1^{(i)}$ and $c_2^{(i)}$ represent the time scales and $q_1^{(i)}$ and $q_2^{(i)}$ characterize magnitudes for the investors preferences for stock i.[6,14] In these definitions, $P_a^{(i)}(t)$ denotes the fundamental value, while $P^{(i)}(t)$ is the trading price of the stock i. Now, from Eqs. (4) and (5) one can obtain the following differential equations for the investor's preferences:

$$\frac{d\zeta_1^{(i)}(t)}{dt} = c_1^{(i)} q_1^{(i)} \frac{1}{P^{(i)}(t)} \frac{dP^{(i)}(t)}{dt} - c_1^{(i)} \zeta_1^{(i)}(t), \tag{6}$$

$$\frac{d\zeta_2^{(i)}(t)}{dt} = c_2^{(i)} q_2^{(i)} \frac{P_a^{(i)}(t) - P^{(i)}(t)}{P_a^{(i)}(t)} - c_2^{(i)} \zeta_2^{(i)}(t), \tag{7}$$

where $i = 1, 2$.

To complete the description of the system, we utilize basic microeconomics principles to define demand and supply functions as follows:

$$D^{(i)} = k^{(i)}(t) M \quad \text{and} \quad S^{(i)} = \widetilde{k}^{(i)}(t) N^{(i)} P^{(i)}(t), \tag{8}$$

where $i = 1, 2$ and M, $N^{(i)}$ are fixed. The price of each stock is then determined by adjustment to the excess demand,[3,14] i.e.,

$$\tau_i \frac{1}{P^{(i)}} \frac{dP^{(i)}}{dt} = F_i\left(\frac{D^{(i)}}{S^{(i)}}\right), \tag{9}$$

where τ_i is the time scale and F_i is an increasing function satisfying $F_i(1) = 0$ for $i = 1, 2$, such as $F_i(x) = x - 1$ or $\log(x)$.

Equations (6)–(9) together with the algebraic equations given in (3) yield a complete dynamical system that can be analyzed qualitatively and solved numerically.

Example: As an example for the equations given in (3), one can take the transition rate functions as follows:

$$\begin{cases} k^{(1)}(t) := \frac{1}{8}\{1 + \tanh[\zeta_1^{(1)}(t) + \zeta_2^{(1)}(t)]\}\{3 + \tanh[-\zeta_1^{(2)}(t) - \zeta_2^{(2)}(t)]\}, \\ k^{(2)}(t) := \frac{1}{8}\{1 + \tanh[\zeta_1^{(2)}(t) + \zeta_2^{(2)}(t)]\}\{3 + \tanh[-\zeta_1^{(1)}(t) - \zeta_2^{(1)}(t)]\}, \\ \widetilde{k}^{(1)}(t) := \frac{1}{2}\{1 - \tanh[\zeta_1^{(1)}(t) + \zeta_2^{(1)}(t)]\}, \\ \widetilde{k}^{(2)}(t) := \frac{1}{2}\{1 - \tanh[\zeta_1^{(2)}(t) + \zeta_2^{(2)}(t)]\}. \end{cases} \tag{10}$$

In this case, the system defined by Eqs. (6)–(9) has the following form:

$$\begin{cases} \tau_1 \dfrac{1}{P^{(1)}}\dfrac{dP^{(1)}}{dt} = F_1\left(\dfrac{M[1+\tanh(\zeta_1^{(1)}+\zeta_2^{(1)})][3+\tanh(-\zeta_1^{(2)}-\zeta_2^{(2)})]}{4N^{(1)}P^{(1)}[1-\tanh(\zeta_1^{(1)}+\zeta_2^{(1)})]}\right), \\ \tau_2 \dfrac{1}{P^{(2)}}\dfrac{dP^{(2)}}{dt} = F_2\left(\dfrac{M[1+\tanh(\zeta_1^{(2)}+\zeta_2^{(2)})][3+\tanh(-\zeta_1^{(1)}-\zeta_2^{(1)})]}{4N^{(2)}P^{(2)}[1-\tanh(\zeta_1^{(2)}+\zeta_2^{(2)})]}\right), \\ \dfrac{d\zeta_1^{(i)}}{dt} = c_1^{(i)}q_1^{(i)}\dfrac{1}{P^{(i)}}\dfrac{dP^{(i)}}{dt} - c_1^{(i)}\zeta_1^{(i)}, \qquad i=1,2, \\ \dfrac{d\zeta_2^{(i)}}{dt} = c_2^{(i)}q_2^{(i)}\dfrac{P_a^{(i)}-P^{(i)}}{P_a^{(i)}} - c_2^{(i)}\zeta_2^{(i)}, \qquad i=1,2. \end{cases} \tag{11}$$

We analyze this system analytically and numerically in later sections.

III. LOCAL STABILITY ANALYSIS OF THE MODEL

For the stability analysis, we first rescale the system defined by Eqs. (6)–(9) under the following constraints and then give a complete stability analysis of the rescaled model for several cases:

(i) $F_1(x) = F_2(x) = x - 1$,
(ii) $P_a^{(i)}(t) = P_a^{(i)} > 0$, $i = 1, 2$, where $P_a^{(i)}$ are constants,
(iii) $c_1^{(i)}$, $c_2^{(i)}$, $q_1^{(i)}$, and $q_2^{(i)}$ are all positive parameters for $i = 1, 2$,
(iv) $\tau_1 = \tau_2 = 1$.

If we rewrite Eqs. (6)–(9) under constraints (i)-(iv), then we have the following system of equations with $i = 1, 2$:

$$\begin{cases} \dfrac{dP^{(i)}}{dt} = \dfrac{k^{(i)}M}{\tilde{k}^{(i)}N^{(i)}} - P^{(i)}, \\ \dfrac{d\zeta_1^{(i)}}{dt} = c_1^{(i)}q_1^{(i)}\dfrac{k^{(i)}M}{\tilde{k}^{(i)}N^{(i)}P^{(i)}} - c_1^{(i)}q_1^{(i)} - c_1^{(i)}\zeta_1^{(i)}, \\ \dfrac{d\zeta_2^{(i)}}{dt} = c_2^{(i)}q_2^{(i)}\left(1-\dfrac{P^{(i)}}{P_a^{(i)}}\right) - c_2^{(i)}\zeta_2^{(i)}. \end{cases} \tag{12}$$

For the stability analysis, we assume that the trading group is affected by only one sentiment while selling or buying each asset. We analyze the stability of system (12) for the following three cases:

1. The trading group has a fixed trading preference.
2. The group has only fundamental trading preferences for both assets.
3. The group follows a mixed trading strategy: A pure value-based strategy while selling or buying the first asset, and a pure trend-based strategy while selling or buying the second asset.

A. The fixed trading preferences

Let us first analyze the dynamics of the system in which $\zeta_1^{(i)}$ and $\zeta_2^{(i)}$, $i = 1, 2$, are assumed to be constant, so $k^{(i)}$ and $\tilde{k}^{(i)}$, $i = 1, 2$, are assumed to be constant. According to this assumption, system (12) is reduced to the following uncoupled system of equations:

$$\frac{dP^{(i)}}{dt} = \frac{k^{(i)}M}{\tilde{k}^{(i)}N^{(i)}} - P^{(i)}, \quad i = 1, 2. \tag{13}$$

The equilibrium point of this system is $[P_{eq}^{(1)}, P_{eq}^{(2)}] = \left(\frac{k^{(1)}M}{\tilde{k}^{(1)}N^{(1)}}, \frac{k^{(2)}M}{\tilde{k}^{(2)}N^{(2)}}\right)$. Equation (13) can be written as $\dot{P}^{(i)} = P_{eq}^{(i)} - P^{(i)}$ which is the linear first order equation. Its solution is

$$P^{(i)}(t) = P_{eq}^{(i)} + [P^{(i)}(0) - P_{eq}^{(i)}]e^{-t}, \quad i = 1, 2. \tag{14}$$

Thus, we have the following result.

Theorem 1. *For the system with the fixed trading preferences governed by (13), the equilibrium point $(P_{eq}^{(1)}, P_{eq}^{(2)})$ is Lyapunov stable and attracting. In other words, it is locally asymptotically stable.*

B. The pure fundamental trading preferences

Now, suppose that the trading group has fundamental trading preferences for both assets, which means that all traders just focus on the deviation between each asset's price and its fundamental value. Then, the transition rate functions can be written as follows:

$$\begin{cases} k^{(1)}(t) = k^{(1)}[\zeta_2^{(1)}(t), \zeta_2^{(2)}(t)], \\ \tilde{k}^{(1)}(t) = \tilde{k}^{(1)}[\zeta_2^{(1)}(t)], \\ k^{(2)}(t) = k^{(2)}[\zeta_2^{(1)}(t), \zeta_2^{(2)}(t)], \\ \tilde{k}^{(2)}(t) = \tilde{k}^{(2)}[\zeta_2^{(2)}(t)], \end{cases} \tag{15}$$

so system (12) is reduced to the following system with $i = 1, 2$:

$$\begin{cases} \dfrac{dP^{(i)}}{dt} = \dfrac{k^{(i)}M}{\tilde{k}^{(i)}N^{(i)}} - P^{(i)}, \\ \dfrac{d\zeta_2^{(i)}}{dt} = c_2^{(i)}q_2^{(i)}\left(1-\dfrac{P^{(i)}}{P_a^{(i)}}\right) - c_2^{(i)}\zeta_2^{(i)}. \end{cases} \tag{16}$$

As a vector equation form, system (16) can be represented as follows:

$$X' = F(X), \tag{17}$$

where $X = (P^{(1)}, P^{(2)}, \zeta_2^{(1)}, \zeta_2^{(2)})^T$, $F = (f_1, f_2, f_3, f_4)^T$ and, for $i = 1, 2$,

$$f_i := \frac{k^{(i)}M}{\tilde{k}^{(i)}N^{(i)}} - P^{(i)},$$

$$f_{i+2} := c_2^{(i)}q_2^{(i)}\left(1-\frac{P^{(i)}}{P_a^{(i)}}\right) - c_2^{(i)}\zeta_2^{(i)}.$$

The equilibrium points of system (17) have the following forms:

$$\begin{aligned} E_F^{eq} &= [P_{eq}^{(1)}, P_{eq}^{(2)}, \zeta_{2,eq}^{(1)}, \zeta_{2,eq}^{(2)}] \\ &= \left[P_{eq}^{(1)}, P_{eq}^{(2)}, q_2^{(1)}\frac{P_a^{(1)}-P_{eq}^{(1)}}{P_a^{(1)}}, q_2^{(2)}\frac{P_a^{(2)}-P_{eq}^{(2)}}{P_a^{(2)}}\right] \\ &= \left[\frac{k^{(1)}M}{\tilde{k}^{(1)}N^{(1)}}, \frac{k^{(2)}M}{\tilde{k}^{(2)}N^{(2)}}, \zeta_{2,eq}^{(1)}, \zeta_{2,eq}^{(2)}\right]. \end{aligned} \tag{18}$$

Notice here that there are two free parameters and the system has infinitely many fixed points.

The Jacobian matrix of system (17) at E_F^{eq} has the form of

$$J(E_F^{eq}) = \begin{bmatrix} -1 & 0 & a & b \\ 0 & -1 & c & d \\ e & 0 & -c_2^{(1)} & 0 \\ 0 & f & 0 & -c_2^{(2)} \end{bmatrix}, \tag{19}$$

where

$$a = \frac{\partial f_1}{\partial \zeta_2^{(1)}}(E_F^{eq}) = \frac{M}{N^{(1)}} \frac{\frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}\widetilde{k}^{(1)} - \frac{\partial \widetilde{k}^{(1)}}{\partial \zeta_2^{(1)}}k^{(1)}}{(\widetilde{k}^{(1)})^2},$$

$$b = \frac{\partial f_1}{\partial \zeta_2^{(2)}}(E_F^{eq}) = \frac{M}{\widetilde{k}^{(1)}2N^{(1)}}\frac{\partial k^{(1)}}{\partial \zeta_2^{(2)}},$$

$$c = \frac{\partial f_2}{\partial \zeta_2^{(1)}}(E_F^{eq}) = \frac{M}{\widetilde{k}^{(2)}N^{(2)}}\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}},$$

$$d = \frac{\partial f_2}{\partial \zeta_2^{(2)}}(E_F^{eq}) = \frac{M}{N^{(2)}} \frac{\frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}}\widetilde{k}^{(2)} - \frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_2^{(2)}}k^{(2)}}{(\widetilde{k}^{(2)})^2},$$

$$e = \frac{\partial f_3}{\partial P^{(1)}}(E_F^{eq}) = \frac{-c_2^{(1)}q_2^{(1)}}{P_a^{(1)}},$$

$$f = \frac{\partial f_4}{\partial P^{(2)}}(E_F^{eq}) = \frac{-c_2^{(2)}q_2^2}{P_a^{(2)}}.$$

Notice that $e < 0$ and $f < 0$ since all parameters $c_2^{(1)}$, $c_2^{(2)}$, $q_2^{(1)}$, $q_2^{(2)}$, and $P_a^{(1)}$, $P_a^{(2)}$ are positive. The characteristic polynomial of $J(E_F^{eq})$ is

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$a_1 = c_2^{(1)} + c_2^{(2)} + 2,$$
$$a_2 = 2c_2^{(1)} + 2c_2^{(2)} + c_2^{(1)}c_2^{(2)} - ea - df + 1,$$
$$a_3 = c_2^{(1)} + c_2^{(2)} + 2c_2^{(1)}c_2^{(2)} - ea - df - eac_2^{(2)} - dfc_2^{(1)},$$
$$a_4 = c_2^{(1)}c_2^{(2)} - eac_2^{(2)} - dfc_2^{(1)} + ef(ad - bc).$$

The Routh-Hurwitz criterion (see Appendix A) states that the roots of the characteristic polynomial have negative real parts if and only if $a_1 > 0$, $a_3 > 0$, $a_4 > 0$, and $a_1a_2a_3 > a_3^2 + a_1^2a_4$.[32,33] The criterion $a_1 > 0$ is satisfied for all positive $c_2^{(1)}$ and $c_2^{(2)}$.

If the following conditions hold:

$$\textbf{C1:}\quad \frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) > 0, \quad \frac{\partial \widetilde{k}^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) < 0,$$

$$\textbf{C2:}\quad \frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) > 0, \quad \frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) < 0,$$

then $a > 0$ and $d > 0$, so criterion $a_3 > 0$ is satisfied since $e < 0$ and $f < 0$. To show that inequality $a_4 > 0$, let us first check whether $ad - bc > 0$ or not,

$$ad - bc = \frac{M^2}{N^{(1)}N^{(2)}(\widetilde{k}^{(1)})^2(\widetilde{k}^{(2)})^2} \times \left[\begin{aligned} & k^{(1)}k^{(2)}\left(\frac{\partial \widetilde{k}^{(1)}}{\partial \zeta_2^{(1)}}\frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_2^{(2)}}\right) \\ & -k^{(1)}\widetilde{k}^{(2)}\left(\frac{\partial \widetilde{k}^{(1)}}{\partial \zeta_2^{(1)}}\frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}}\right) \\ & -\widetilde{k}^{(1)}k^{(2)}\left(\frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}\frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_2^{(2)}}\right) \\ & +\widetilde{k}^{(1)}\widetilde{k}^{(2)}\left(\frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}\frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}} - \frac{\partial k^{(1)}}{\partial \zeta_2^{(2)}}\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}\right) \end{aligned} \right].$$

If the following conditions hold:

$$\textbf{C3:}\quad \frac{\partial k^{(1)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) < 0, \quad \frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) < 0,$$

$$\textbf{C4:}\quad \frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq})\frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) > \frac{\partial k^{(1)}}{\partial \zeta_2^{(2)}}(E_F^{eq})\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_F^{eq}),$$

then $ad - bc > 0$. Eventually, if conditions **C1**, **C2**, **C3**, and **C4** hold, then criterion $a_4 > 0$ is satisfied since $e < 0$ and $f < 0$. The final criterion $a_1a_2a_3 > a_3^2 + a_1^2a_4$ is equivalent to

$$\begin{aligned} & a_1a_2a_3 - a_3^2 - a_1^2a_4 \\ & = -fd[3(c_2^{(1)})^2c_2^{(2)} + 4c_2^{(1)}(c_2^{(2)})^2 + 7c_2^{(1)}c_2^{(2)} + 2(c_2^{(1)})^2 + 3c_2^{(1)} \\ & \quad + 3(c_2^{(2)})^2 + 5c_2^{(2)} + (c_2^{(1)})^3c_2^{(2)} + (c_2^{(1)})^2(c_2^{(2)})^2 + 2(c_2^{(1)})^3 + 2] \\ & \quad - ae[(c_2^{(1)})^2(c_2^{(2)})^2 + 3(c_2^{(1)})^2c_2^{(2)} + 7c_2^{(1)}c_2^{(2)} + 3(c_2^{(1)})^2 + 5c_2^{(1)} \\ & \quad + 2(c_2^{(2)})^2 + 3c_2^{(2)} + (c_2^{(1)})^2(c_2^{(2)})^2 + c_2^{(1)}(c_2^{(2)})^3 + 2(c_2^{(2)})^3 + 2] \\ & \quad + (ae - fd)^2\left[c_2^{(1)} + c_2^{(1)}c_2^{(2)} + c_2^{(2)} + 1\right] \\ & \quad + efbc\left[(c_2^{(1)})^2 + (c_2^{(2)})^2 + 2c_2^{(1)}c_2^{(2)} + 4c_2^{(1)} + 4c_2^{(2)} + 4\right] \\ & \quad + 2(c_2^{(1)})^3(c_2^{(2)})^2 + 2(c_2^{(2)})^3(c_2^{(1)})^2 + 8(c_2^{(1)})^2(c_2^{(2)})^2 + 4(c_2^{(1)})^3c_2^{(2)} \\ & \quad + 10(c_2^{(1)})^2c_2^{(2)} + 4c_2^{(1)}(c_2^{(2)})^3 + 10c_2^{(1)}(c_2^2)^2 + 8c_2^{(1)}c_2^{(2)} + 2(c_2^{(1)})^3 \\ & \quad + 4(c_2^{(1)})^2 + 2c_2^{(1)} + 2(c_2^{(2)})^3 + 4(c_2^{(2)})^2 + 2c_2^{(2)}. \end{aligned}$$

Note that if conditions **C1**, **C2**, and **C3** hold, then $fd < 0$, $ae < 0$, and $efbc > 0$. So, the final criterion $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$ is satisfied for all positive $c_2^{(1)}$ and $c_2^{(2)}$. We have just validated the following result.

Theorem 2. *The equilibrium point (18) of system (16) is asymptotically stable if the following conditions hold:*

$$\textbf{C1}:\quad \frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) > 0, \quad \frac{\partial \widetilde{k}^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) < 0,$$

$$\textbf{C2}:\quad \frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) > 0, \quad \frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) < 0,$$

$$\mathbf{C3}: \frac{\partial k^{(1)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) < 0, \quad \frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) < 0,$$

$$\mathbf{C4}: \frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq})\frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) > \frac{\partial k^{(1)}}{\partial \zeta_2^{(2)}}(E_F^{eq})\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_F^{eq}).$$

In plain language, Theorem 2 says that in a group of investors that base their decisions on the value of a stock and not its price movement, the equilibrium is stable provided the following conditions are satisfied: **(C1)** the likelihood of buying stock 1 by the group increases with the stock sentiment (which increases with decreasing price) and the likelihood of selling stock 1 decreases with its sentiment, **(C2)** similar holds for stock 2, **(C3)** the likelihood of buying stock 1 decreases with increasing sentiment for stock 2 and vice versa, and **(C4)** the increase in the likelihood of purchasing stocks 1 and 2 based on their own sentiments exceeds the combined decrease in the likelihood of their purchase based on the opposite stock sentiments. The last condition can be guaranteed by requiring that the influence of each stock sentiment on its own stock trading rate is larger than its influence on the trading rate of the other stock.

C. The mixed trading preferences

We now consider a system in which the trading group has a different strategy for each stock. We assume that the group follows a pure value-based strategy for the first stock but follows a pure trend-based strategy for the second stock, so the transition rate functions are defined as follows:

$$\begin{cases} k^{(1)}(t) = k^{(1)}[\zeta_2^{(1)}(t), \zeta_1^{(2)}(t)], \\ \tilde{k}^{(1)}(t) = \tilde{k}^{(1)}[\zeta_2^{(1)}(t)], \\ k^{(2)}(t) = k^{(2)}[\zeta_2^{(1)}(t),\ \zeta_1^{(2)}(t)], \\ \tilde{k}^{(2)}(t) = \tilde{k}^{(2)}[\zeta_1^{(2)}(t)]. \end{cases} \tag{20}$$

Then, system (12) is reduced to the following system:

$$\begin{cases} \dfrac{dP^{(1)}}{dt} = \dfrac{k^{(1)}M}{\tilde{k}^{(1)}N^{(1)}} - P^{(1)}, \\ \dfrac{dP^{(2)}}{dt} = \dfrac{k^{(2)}M}{\tilde{k}^{(2)}N^{(2)}} - P^{(2)}, \\ \dfrac{d\zeta_2^{(1)}}{dt} = c_2^{(1)}q_2^{(1)}\dfrac{P_a^{(1)} - P^{(1)}}{P_a^{(1)}} - c_2^{(1)}\zeta_2^{(1)}, \\ \dfrac{d\zeta_1^{(2)}}{dt} = c_1^{(2)}q_1^{(2)}\dfrac{k^{(2)}M}{\tilde{k}^{(2)}N^{(2)}P^{(2)}} - c_1^{(2)}q_1^{(2)} - c_1^{(2)}\zeta_1^{(2)}. \end{cases} \tag{21}$$

Representing system (21) as a vector equation form, one has

$$X' = F(X), \tag{22}$$

where $X = [P^{(1)}, P^{(2)}, \zeta_2^{(1)}, \zeta_1^{(2)}]^T$, $F = (f_1 f_2, f_3, f_4)^T$ and

$$f_1 := \frac{k^{(1)}M}{\tilde{k}^{(1)}N^{(1)}} - P^{(1)},$$

$$f_2 := \frac{k^{(2)}M}{\tilde{k}^{(2)}N^{(2)}} - P^{(2)},$$

$$f_3 := c_2^{(1)}q_2^{(1)}\frac{P_a^{(1)} - P^{(1)}}{P_a^{(1)}} - c_2^{(1)}\zeta_2^{(1)},$$

$$f_4 := c_1^{(2)}q_1^{(2)}\frac{k^{(2)}M}{\tilde{k}^{(2)}N^{(2)}P^{(2)}} - c_1^{(2)}q_1^{(2)} - c_1^{(2)}\zeta_1^{(2)}.$$

The equilibrium points of system (22) can be obtained by solving the following equation for $P^{(1)}$, $P^{(2)}$, $\zeta_2^{(1)}$, $\zeta_1^{(2)}$:

$$F(X) = 0.$$

$f_1 = 0$ and $f_2 = 0$ yield

$$P_{eq}^{(1)} = \frac{k^{(1)}M}{\tilde{k}^{(1)}N^{(1)}}, \tag{23}$$

$$P_{eq}^{(2)} = \frac{k^{(2)}M}{\tilde{k}^{(2)}N^{(2)}}. \tag{24}$$

From $f_4 = 0$ together with Eq. (24), we obtain $\zeta_{1,eq}^{(2)} = 0$. Finally, from $f_3 = 0$, we have

$$\zeta_{2,eq}^{(1)} = q_2^{(1)}\frac{P_a^{(1)} - P_{eq}^{(1)}}{P_a^{(1)}}. \tag{25}$$

The equilibrium points of system (22) have the following forms:

$$E_M^{eq} = [P_{eq}^{(1)}, P_{eq}^{(2)}, \zeta_{2,eq}^{(1)}, \zeta_{1,eq}^{(2)}] = \left(P_{eq}^{(1)}, P_{eq}^{(2)}, q_2^{(1)}\frac{P_a^{(1)} - P_{eq}^{(1)}}{P_a^{(1)}}, 0\right). \tag{26}$$

Once again, the system has infinitely many equilibrium points. The Jacobian matrix of system (22) at E_M^{eq} has the form of

$$J(E_M^{eq}) = \begin{bmatrix} -1 & 0 & a & b \\ 0 & -1 & c & d \\ f & 0 & -c_2^{(1)} & 0 \\ 0 & e & -ec & -ed - c_1^{(2)} \end{bmatrix}, \tag{27}$$

where

$$a = \frac{\partial f_1}{\partial \zeta_2^{(1)}}(E_M^{eq}) = \frac{M}{N^{(1)}}\frac{\frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}\tilde{k}^{(1)} - \frac{\partial \tilde{k}^{(1)}}{\partial \zeta_2^{(1)}}k^{(1)}}{(\tilde{k}^{(1)})^2},$$

$$b = \frac{\partial f_1}{\partial \zeta_1^{(2)}}(E_M^{eq}) = \frac{M}{\tilde{k}^{(1)}N^{(1)}}\frac{\partial k^{(1)}}{\partial \zeta_1^{(2)}},$$

$$c = \frac{\partial f_2}{\partial \zeta_2^{(1)}}(E_M^{eq}) = \frac{M}{\tilde{k}^{(2)}N^{(2)}}\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}},$$

$$d = \frac{\partial f_2}{\partial \zeta_1^{(2)}}(E_M^{eq}) = \frac{M}{N^{(2)}} \frac{\frac{\partial k^{(2)}}{\partial \zeta_1^{(2)}}\widetilde{k}^{(2)} - \frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_1^{(2)}} k^{(2)}}{(\widetilde{k}^{(2)})^2},$$

$$e = \frac{\partial f_4}{\partial P^{(2)}}(E_M^{eq}) = -\frac{c_1^{(2)} q_1^{(2)}}{P_{eq}^{(2)}},$$

$$f = \frac{\partial f_3}{\partial P^{(1)}}(E_M^{eq}) = -\frac{c_2^{(1)} q_2^{(1)}}{P_a^{(1)}}.$$

The characteristic polynomial of $J(E_M^{eq})$ is

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$a_1 = 2 + de + c_1^{(2)} + c_2^{(1)},$$
$$a_2 = 2c_1^{(2)} + 2c_2^{(1)} + c_1^{(2)}c_2^{(1)} + ed - af + edc_2^{(1)} + 1,$$
$$a_3 = c_1^{(2)} + c_2^{(1)} + 2c_1^{(2)}c_2^{(1)} - af + edc_2^{(1)} - afc_1^{(2)} - eadf + ebcf,$$
$$a_4 = c_1^{(2)}c_2^{(1)} - afc_1^{(2)}.$$

Using the Routh-Hurwitz criteria, we now determine the conditions for local stability. Let us define $U = c_1^{(2)} + de$, then

$$a_1 = c_2^{(1)} + U + 2,$$
$$a_2 = c_1^{(2)} + 2c_2^{(1)} + c_2^{(1)}U + U - af + 1,$$
$$a_3 = c_1^{(2)} + c_2^{(1)} + c_1^{(2)}c_2^{(1)} + c_2^{(1)}U - af - afU + ebcf,$$
$$a_4 = c_1^{(2)}c_2^{(1)} - afc_1^{(2)}.$$

If the following condition holds:

K1: $\frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) > 0,\ \frac{\partial k^{(1)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) < 0,\ \frac{\partial \widetilde{k}^{(1)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) < 0,$

$\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) < 0,\ \frac{\partial k^{(2)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) > 0,\ \frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) < 0,$

then $a > 0,\ b < 0,\ c < 0,\ d > 0$.

Note that $e < 0,\ f < 0$ since all parameters are positive, and $P_a^{(1)} > 0,\ P_{eq}^{(2)} > 0$. Hence, if condition **K1** and the following condition hold:

K2: $U > 0,$

then $a_1 > 0,\ a_3 > 0,\ a_4 > 0$.

Now, define $V = c_2^{(1)} - af,\ Y = ebcf,\ Z = c_2^{(1)} + 1$. If condition **K1** holds, then $V > 0,\ Y > 0$, and $Z > 0$.

So, $a_1,\ a_2,\ a_3,\ a_4$, and the fourth inequality $a_1a_2a_3 > a_3^2 + a_1^2a_4$ can be rewritten as

$$a_1 = U + Z + 1,$$
$$a_2 = c_1^{(2)} + Z + V + UZ,$$
$$a_3 = c_1^{(2)}Z + V + UV + Y,$$
$$a_4 = c_1^{(2)}V,$$

$$\begin{aligned}
a_1a_2a_3 - a_3^2 - a_1^2a_4 &= U^3VZ + U^2VZ^2 + 3U^2VZ + U^2YZ \\
&+ U^2Z^2c_1^{(2)} + UV^2Z - UVY + 2UVZ^2 \\
&- 2UVZc_1^{(2)} + 3UVZ + UYZ^2 + 2UYZ \\
&+ UYc_1^{(2)} + UZ^3c_1^{(2)} + 2UZ^2c_1^{(2)} + UZ(c_1^{(2)})^2 \\
&+ V^2Z + VYZ - VY + VZ^2 - 2VZc_1^{(2)} \\
&+ VZ - Y^2 + YZ^2 - YZc_1^{(2)} \\
&+ YZ + Yc_1^{(2)} + Z^3c + Z^2c_1^{(2)} + Z(c_1^{(2)})^2 \\
&= 2UYZ - UVY + YZ - VY \\
&+ 2UVZ^2 - 2UVZc_1^{(2)} + VZ^2 - 2VZc_1^{(2)} \\
&+ Yc_1^{(2)} - Y^2 + Z(c_1^{(2)})^2 - YZc_1^{(2)} \\
&+ [\text{other terms that are positive}] \\
&= \underbrace{2UVZ(Z - c_1^{(2)})}_{T_1} + \underbrace{VZ(Z - 2c_1^{(2)})}_{T_2} \\
&+ \underbrace{UY(2Z - V)}_{T_3} + \underbrace{Y(Z - V)}_{T_4} \\
&+ \underbrace{Y(c_1^{(2)} - Y)}_{T_5} + \underbrace{Zc_1^{(2)}(c_1^{(2)} - Y)}_{T_6} \\
&+ [\text{other terms that are positive}].
\end{aligned}$$

- If $Z - 2c_1^{(2)} = c_2^{(1)} + 1 - 2c_1^{(2)} > 0$, then $T_1 > 0$ (since, if $Z - 2c_1^{(2)} > 0$, then $Z - c_1^{(2)} > 0$) and $T_2 > 0$.
- If $Z - V = c_2^{(1)} + 1 - c_2^{(1)} + af = 1 + af > 0$, then $T_3 > 0$ (since, if $Z - V > 0$, then $2Z - V > 0$) and $T_4 > 0$.
- If $c_1^{(2)} - Y = c_1^{(2)} - bcef = c_1^{(2)} - bc\frac{c_2^{(1)}q_2^{(1)}}{P_a^{(1)}}\frac{c_1^{(2)}q_1^{(2)}}{P_{eq}^{(2)}} = c_1^{(2)}\left(1 - bc\frac{c_2^{(1)}q_2^{(1)}}{P_a^{(1)}}\frac{q_1^{(2)}}{P_{eq}^{(2)}}\right) > 0$, then $T_5 > 0$ and $T_6 > 0$.

Consequently, if conditions **K1**, **K2**, and the following conditions are satisfied:

K3: $c_2^{(1)} + 1 - 2c_1^{(2)} > 0,$

K4: $1 + af = 1 - a\frac{c_2^{(1)}q_2^{(1)}}{P_a^{(1)}} > 0,$

K5: $1 - bc\frac{c_2^{(1)}q_2^{(1)}}{P_a^{(1)}}\frac{q_1^{(2)}}{P_{eq}^{(2)}} > 0,$

then $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$. Thus, we reach the following result.

Theorem 3. *The equilibrium point E_M^{eq} of system (21) in which all traders follow a pure value-based strategy while selling or buying asset 1 and a pure trend-based strategy while selling or buying asset 2 is asymptotically stable if the following* *conditions hold:*

$$\begin{aligned}
\mathbf{K1}: &\quad \frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) > 0, \quad \frac{\partial k^{(1)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) < 0, \quad \frac{\partial \widetilde{k}^{(1)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) < 0,\\
&\quad \frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) < 0, \quad \frac{\partial k^{(2)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) > 0, \quad \frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) < 0,\\
\mathbf{K2}: &\quad 1 - \frac{M q_1^{(2)}}{N^{(2)} P_{eq}^{(2)}} \frac{\partial}{\partial \zeta_1^{(2)}} \left(\frac{k^{(2)}}{\widetilde{k}^{(2)}} \right) > 0,\\
\mathbf{K3}: &\quad c_2^{(1)} + 1 - 2c_1^{(2)} > 0,\\
\mathbf{K4}: &\quad 1 - \frac{M c_2^{(1)} q_2^{(1)}}{N^{(1)} P_a^{(1)}} \frac{\partial}{\partial \zeta_2^{(1)}} \left(\frac{k^{(1)}}{\widetilde{k}^{(1)}} \right) > 0,\\
\mathbf{K5}: &\quad 1 - \frac{M^2 c_2^{(1)} q_2^{(1)} q_1^{(2)}}{N^{(1)} N^{(2)} P_a^{(1)} P_{eq}^{(2)}} \frac{\left(\frac{\partial k^{(1)}}{\partial \zeta_1^{(2)}} \right) \left(\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}} \right)}{\widetilde{k}^{(1)} \widetilde{k}^{(2)}} > 0.
\end{aligned}$$

In plain language, Theorem 3 states that in a group of investors that base their decisions on the value of stock 1 and price increase of stock 2, the equilibrium is stable provided the following conditions are satisfied: (**K1**) the likelihood of buying stock 1 by the group increases with that stock value sentiment (which increases with decreasing price) but decreases with stock 2 trend sentiment, and the likelihood of selling stock 1 decreases with its value sentiment, while the likelihood of buying stock 2 decreases with the value sentiment of stock 1, increases with the trend sentiment of stock 2 and the likelihood of selling stock 2 decreases with its trend sentiment, (**K2**) the change in the ratio of the likelihood of buying versus selling of stock 2 cannot depend much on that stock trend sentiment, (**K3**) the trend-based sentiment for stock 2 must react slowly to changes in price, (**K4**) the change in the ratio of the likelihood of buying versus selling of stock 1 cannot depend too much on that stock value sentiment, and (**K5**) the combined influence of stock 1 value sentiment on stock 2 purchase likelihood and the influence of stock 2 trend sentiment on stock 1 purchase likelihood must be small.

IV. EXAMPLE AND NUMERICAL SIMULATIONS

As an example, we consider system (11) in which the transition rate function is defined by (10). We rewrite the system under the following constraints for stability analysis:

(i) $F_1(x) = F_2(x) = x - 1$,
(ii) $|\zeta_1^{(1)}(t) + \zeta_2^{(1)}(t)| < \epsilon_1$ and $|\zeta_1^{(2)}(t) + \zeta_2^{(2)}(t)| < \epsilon_2$, where ϵ_1 and ϵ_2 are small positive numbers,
(iii) to simplify the calculations, we use the Taylor series approximation of tanh function, i.e., $\tanh(x) \simeq x$. Thus, the transition rate functions can be written as follows [due to assumptions (ii) and (iii)]:

$$\begin{cases}
k^{(1)}(t) \approx \dfrac{1}{8}[1 + \zeta_1^{(1)}(t) + \zeta_2^{(1)}(t)][3 - \zeta_1^{(2)}(t) - \zeta_2^{(2)}(t)],\\
k^{(2)}(t) \approx \dfrac{1}{8}[1 + \zeta_1^{(2)}(t) + \zeta_2^{(2)}(t)][3 - \zeta_1^{(1)}(t) - \zeta_2^{(1)}(t)],\\
\widetilde{k}^{(1)}(t) \approx \dfrac{1}{2}[1 - \zeta_1^{(1)}(t) - \zeta_2^{(1)}(t)],\\
\widetilde{k}^{(2)}(t) \approx \dfrac{1}{2}[1 - \zeta_1^{(2)}(t) - \zeta_2^{(2)}(t)],
\end{cases} \tag{28}$$

(iv) $P_a^{(1)}(t) = P_a^{(1)} > 0$ and $P_a^{(2)}(t) = P_a^{(2)} > 0$, where both $P_a^{(1)}$ and $P_a^{(2)}$ are constants,
(v) $c_1^{(i)}$, $c_2^{(i)}$, $q_1^{(i)}$, and $q_2^{(i)}$ are all positive parameters for $i = 1, 2$,
(vi) $\tau_1 = \tau_2 = 1$.

Under the above constraints, system (11) turns into the following system of equations:

$$\begin{cases}
\dfrac{dP^{(1)}}{dt} = \dfrac{M(1 + \zeta_1^{(1)} + \zeta_2^{(1)})(3 - \zeta_1^{(2)} - \zeta_2^{(2)})}{4N^{(1)}(1 - \zeta_1^{(1)} - \zeta_2^{(1)})} - P^{(1)},\\
\dfrac{dP^{(2)}}{dt} = \dfrac{M(1 + \zeta_1^{(2)} + \zeta_2^{(2)})(3 - \zeta_1^{(1)} - \zeta_2^{(1)})}{4N^{(2)}(1 - \zeta_1^{(2)} - \zeta_2^{(2)})} - P^{(2)},\\
\dfrac{d\zeta_1^{(1)}}{dt} = c_1^{(1)} q_1^{(1)} \dfrac{M(1 + \zeta_1^{(1)} + \zeta_2^{(1)})(3 - \zeta_1^{(2)} - \zeta_2^{(2)})}{4N^{(1)}(1 - \zeta_1^{(1)} - \zeta_2^{(1)})P^{(1)}}\\
\qquad - c_1^{(1)} q_1^{(1)} - c_1^{(1)} \zeta_1^{(1)},\\
\dfrac{d\zeta_2^{(1)}}{dt} = c_2^{(1)} q_2^{(1)} \left(1 - \dfrac{P^{(1)}}{P_a^{(1)}}\right) - c_2^{(1)} \zeta_2^{(1)},\\
\dfrac{d\zeta_1^{(2)}}{dt} = c_1^{(2)} q_1^{(2)} \dfrac{M(1 + \zeta_1^{(2)} + \zeta_2^{(2)})(3 - \zeta_1^{(1)} - \zeta_2^{(1)})}{4N^{(2)}(1 - \zeta_1^{(2)} - \zeta_2^{(2)})P^{(2)}}\\
\qquad - c_1^{(2)} q_1^{(2)} - c_1^{(2)} \zeta_1^{(2)},\\
\dfrac{d\zeta_2^{(2)}}{dt} = c_2^{(2)} q_2^{(2)} \left(1 - \dfrac{P^{(2)}}{P_a^{(2)}}\right) - c_2^{(2)} \zeta_2^{(2)}.
\end{cases} \tag{29}$$

Notice here that system (29) yields an example for system (12). Once again, we assume that the trading group is affected by only one sentiment while selling or buying each asset and analyze the stability of system (29) for the following two cases:

Case 1: The group has fundamental trading preferences for each asset,

Case 2: The group follows a mixed trading preference for each asset: A pure value-based strategy while selling or buying the first asset, and a pure trend-based strategy while selling or buying the second asset.

A. Case 1. The pure fundamental trading preferences

Suppose that the trading group follows a pure value-based strategy for each asset, i.e., all traders focus on only the deviation between the asset price and its fundamental value and ignore the trend for trading. Then, the transition rate functions (28) can be written as follows:

$$\begin{cases}
k^{(1)} = \dfrac{1}{8}[1 + \zeta_2^{(1)}][3 - \zeta_2^{(2)}],\\
\widetilde{k}^{(1)} = \dfrac{1}{2}[1 - \zeta_2^{(1)}],\\
k^{(2)} = \dfrac{1}{8}[1 + \zeta_2^{(2)}][3 - \zeta_2^{(1)}],\\
\widetilde{k}^{(2)} = \dfrac{1}{2}[1 - \zeta_2^{(2)}].
\end{cases} \tag{30}$$

As a result, system (29) turns into the following system:

$$\begin{cases} \dfrac{dP^{(1)}}{dt} = \dfrac{M(1+\zeta_2^{(1)})(3-\zeta_2^{(2)})}{4N^{(1)}(1-\zeta_2^{(1)})} - P^{(1)}, \\ \dfrac{dP^{(2)}}{dt} = \dfrac{M(1+\zeta_2^{(2)})(3-\zeta_2^{(1)})}{4N^{(2)}(1-\zeta_2^{(2)})} - P^{(2)}, \\ \dfrac{d\zeta_2^{(1)}}{dt} = c_2^{(1)} q_2^{(1)} \dfrac{P_a^{(1)} - P^{(1)}}{P_a^{(1)}} - c_2^{(1)}\zeta_2^{(1)}, \\ \dfrac{d\zeta_2^{(2)}}{dt} = c_2^{(2)} q_2^{(2)} \dfrac{P_a^{(2)} - P^{(2)}}{P_a^{(2)}} - c_2^{(2)}\zeta_2^{(2)}, \end{cases} \tag{31}$$

and its equilibrium points has the following form:

$$E_F^{eq} = [P_{eq}^{(1)}, P_{eq}^{(2)}, \zeta_{2,eq}^{(1)}, \zeta_{2,eq}^{(2)}] = \left(\frac{(1+\zeta_{2,eq}^{(1)})(3-\zeta_{2,eq}^{(2)})M}{4(1-\zeta_{2,eq}^{(1)})N^{(1)}}, \frac{(1+\zeta_{2,eq}^{(2)})(3-\zeta_{2,eq}^{(1)})M}{4(1-\zeta_{2,eq}^{(2)})N^{(2)}}, \zeta_{2,eq}^{(1)}, \zeta_{2,eq}^{(2)} \right). \tag{32}$$

Following now Eq. (19), one obtains the characteristic polynomial of $J(E_F^{eq})$ as follows:

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$\begin{aligned} a_1 &= c_2^{(1)} + c_2^{(2)} + 2, \\ a_2 &= 2c_2^{(1)} + 2c_2^{(2)} + c_2^{(1)}c_2^{(2)} - ea - df + 1, \\ a_3 &= c_2^{(1)} + c_2^{(2)} + 2c_2^{(1)}c_2^{(2)} - ea - df - eac_2^{(2)} - dfc_2^{(1)}, \\ a_4 &= c_2^{(1)}c_2^{(2)} - eac_2^{(2)} - dfc_2^{(1)} + ef(ad - bc), \end{aligned}$$

in which

$$a = \frac{M(3-\zeta_{2,eq}^{(2)})}{2N^{(1)}(1-\zeta_{2,eq}^{(1)})^2},\ b = \frac{-M(1+\zeta_{2,eq}^{(1)})}{4N^{(1)}(1-\zeta_{2,eq}^{(1)})},\ c = \frac{-M(1+\zeta_{2,eq}^{(2)})}{4N^{(2)}(1-\zeta_{2,eq}^{(2)})},$$

$$d = \frac{M(3-\zeta_{2,eq}^{(1)})}{2N^{(2)}(1-\zeta_{2,eq}^{(2)})^2},\ e = \frac{-c_2^{(1)}q_2^{(1)}}{P_a^{(1)}},\ f = \frac{-c_2^{(2)}q_2^{(2)}}{P_a^{(2)}}.$$

Now, let us check conditions **C1**, **C2**, **C3**, and **C4** given in Sec. III B. First, note that since it is assumed that $\tanh(x) \simeq x$, we have $-1 < \zeta_2^{(1)} < 1$ and $-1 < \zeta_2^{(2)} < 1$. So, the following inequalities are satisfied:

$$0 < 1 - \zeta_2^{(1)} < 2, \tag{33}$$

$$0 < 1 + \zeta_2^{(1)} < 2, \tag{34}$$

$$2 < 3 - \zeta_2^{(1)} < 4, \tag{35}$$

$$0 < 1 - \zeta_2^{(2)} < 2, \tag{36}$$

$$0 < 1 + \zeta_2^{(2)} < 2, \tag{37}$$

$$2 < 3 - \zeta_2^{(2)} < 4. \tag{38}$$

- According to inequality (38),
$$\frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) = \frac{1}{8}[3 - \zeta_{2,eq}^{(2)}] > 0$$
and
$$\frac{\partial \tilde{k}^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) = -\frac{1}{2} < 0.$$
Thus, condition **C1** holds.
- According to inequality (35),
$$\frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) = \frac{1}{8}[3 - \zeta_{2,eq}^{(1)}] > 0$$
and
$$\frac{\partial \tilde{k}^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) = -\frac{1}{2} < 0.$$
Therefore, condition **C2** is satisfied.
- According to inequalities (34) and (37),
$$\frac{\partial k^{(1)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) = -\frac{1}{8}[1 + \zeta_{2,eq}^{(1)}] < 0,$$
$$\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) = -\frac{1}{8}(1 + \zeta_{2,eq}^{(2)}) < 0$$
so that condition **C3** holds.
- Finally, we show that condition **C4** is satisfied. First, it is easy to see that
$$\begin{aligned} &\frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_F^{eq})\frac{\partial k^{(2)}}{\partial \zeta_2^{(2)}}(E_F^{eq}) - \frac{\partial k^{(1)}}{\partial \zeta_2^{(2)}}(E_F^{eq})\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_F^{eq}) \\ &= \frac{1}{8}(3 - \zeta_{2,eq}^{(2)})\frac{1}{8}(3 - \zeta_{2,eq}^{(1)}) - \frac{1}{8}(1 + \zeta_{2,eq}^{(1)})\frac{1}{8}(1 + \zeta_{2,eq}^{(2)}) \\ &= 1 - \frac{1}{2}(\zeta_{2,eq}^{(1)} + \zeta_{2,eq}^{(2)}). \end{aligned}$$
Since $-1 < \zeta_2^{(1)} < 1$ and $-1 < \zeta_2^{(2)} < 1$, one has $-2 < \zeta_2^{(1)} + \zeta_2^{(2)} < 2$. Thus,
$$1 - \frac{1}{2}[\zeta_{2,eq}^{(1)} + \zeta_{2,eq}^{(2)}] > 0,$$
so that condition **C4** holds.

Consequently, since system (31) is an example of system (16), according to Theorem 2, we have proved the following result.

Corollary 1. *The equilibrium point (32) of system (31) is asymptotically stable for all positive* $c_2^{(1)}$ *and* $c_2^{(2)}$.

B. Case 2. The mixed trading preferences

1. Local stability analysis

We now assume that the trading group has different strategies for each stock, i.e., the group follows a pure value-based strategy for the first stock but follows a pure trend-based strategy for the second stock. Hence, the transition rate functions (28) are given by

$$\begin{cases} k^{(1)} = \dfrac{1}{8}[1+\zeta_2^{(1)}][3-\zeta_1^{(2)}], \\ \widetilde{k}^{(1)} = \dfrac{1}{2}[1-\zeta_2^{(1)}], \\ k^{(2)} = \dfrac{1}{8}[1+\zeta_2^{(1)}][3-\zeta_1^{(2)}], \\ \widetilde{k}^{(2)} = \dfrac{1}{2}[1-\zeta_1^{(2)}]. \end{cases} \tag{39}$$

Thus, system (29) is reduced to the following form:

$$\begin{cases} \dfrac{dP^{(1)}}{dt} = \dfrac{M(1+\zeta_2^{(1)})(3-\zeta_1^{(2)})}{4N^{(1)}(1-\zeta_2^{(1)})} - P^{(1)}, \\ \dfrac{dP^{(2)}}{dt} = \dfrac{M(1+\zeta_1^{(2)})(3-\zeta_2^{(1)})}{4N^{(2)}(1-\zeta_1^{(2)})} - P^{(2)}, \\ \dfrac{d\zeta_2^{(1)}}{dt} = c_2^{(1)}q_2^{(1)}\dfrac{P_a^{(1)}-P^{(1)}}{P_a^{(1)}} - c_2^{(1)}\zeta_2^{(1)}, \\ \dfrac{d\zeta_1^{(2)}}{dt} = c_1^{(2)}q_1^{(2)}\dfrac{M}{4N^{(2)}P^{(2)}}\dfrac{(1+\zeta_1^{(2)})(3-\zeta_2^{(1)})}{(1-\zeta_1^{(2)})} - c_1^{(2)}q_1^{(2)} - c_1^{(2)}\zeta_1^{(2)}. \end{cases} \tag{40}$$

Using Eqs. (23)–(25), one can obtain the equilibrium points of system (40) as follows:

$$P_{eq}^{(1)} = \frac{3M}{4N^{(1)}}\frac{1+\zeta_{2,eq}^{(1)}}{1-\zeta_{2,eq}^{(1)}}, \tag{41}$$

$$P_{eq}^{(2)} = \frac{M}{4N^{(2)}}[3-\zeta_{2,eq}^{(1)}], \tag{42}$$

$$\zeta_{1,eq}^{(2)} = 0, \tag{43}$$

$$\zeta_{2,eq}^{(1)} = q_2^{(1)}\frac{P_a^{(1)}-P_{eq}^{(1)}}{P_a^{(1)}}. \tag{44}$$

Now, combining Eqs. (41) and (44) we have the following equation:

$$\frac{q_2^{(1)}}{P_a^{(1)}}[P_{eq}^{(1)}]^2 + \left(1-q_2^{(1)}+q_2^{(1)}\frac{3M}{4N^{(1)}P_a^{(1)}}\right)P_{eq}^{(1)} - \frac{3M}{4N^{(1)}}(1+q_2^{(1)}) = 0. \tag{45}$$

Solving Eq. (45) for $P_{eq}^{(1)}$ yields the following positive root:

$$P_{eq}^{(1)} = \frac{-\left(1-q_2^{(1)}+q_2^{(1)}\frac{3M}{4N^{(1)}P_a^{(1)}}\right)+\sqrt{\left(1-q_2^{(1)}+q_2^{(1)}\frac{3M}{4N^{(1)}P_a^{(1)}}\right)^2+\frac{q_2^{(1)}}{P_a^{(1)}}\frac{3M}{N^{(1)}}(1+q_2^{(1)})}}{\frac{2q_2^{(1)}}{P_a^{(1)}}}. \tag{46}$$

Since $P^{(1)}$ is the price of the first stock, it cannot be negative. Therefore, we omitted the negative root of Eq. (45). The equilibrium points of system (40) have the following forms:

$$\begin{aligned} E_M^{eq} &= (P_{eq}^{(1)}, P_{eq}^{(2)}, \zeta_{2,eq}^{(1)}, \zeta_{1,eq}^{(2)}) \\ &= \left(P_{eq}^{(1)}, \frac{M}{4N^{(2)}}\left(3 - q_2^{(1)}\frac{P_a^{(1)} - P_{eq}^{(1)}}{P_a^{(1)}}\right), q_2^{(1)}\frac{P_a^{(1)} - P_{eq}^{(1)}}{P_a^{(1)}}, 0\right) \\ &= \left(\frac{3M(1+\zeta_{2,eq}^{(1)})}{4N^{(1)}(1-\zeta_{2,eq}^{(1)})}, \frac{M(3-\zeta_{2,eq}^{(1)})}{4N^{(2)}}, \zeta_{2,eq}^{(1)}, 0\right) \end{aligned} \tag{47}$$

in which $P_{eq}^{(1)}$ is the equilibrium price which is denoted by Eq. (46). From (27), the characteristic polynomial of $J(E_M^{eq})$ can be obtained as follows:

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$\begin{aligned} a_1 &= 2 + de + c_1^{(2)} + c_2^{(1)}, \\ a_2 &= 2c_1^{(2)} + 2c_2^{(1)} + c_1^{(2)}c_2^{(1)} + ed - af + edc_2^{(1)} + 1, \\ a_3 &= c_1^{(2)} + c_2^{(1)} + 2c_1^{(2)}c_2^{(1)} - af + edc_2^{(1)} - afc_1^{(2)} \\ &\quad - eadf + ebcf, \\ a_4 &= c_1^{(2)}c_2^{(1)} - afc_1^{(2)} \end{aligned}$$

in which $a = \frac{3M}{2N^{(1)}}\frac{1}{(1-\zeta_{2,eq}^{(1)})^2}$, $b = \frac{-M}{4N^{(1)}}\frac{(1+\zeta_{2,eq}^{(1)})}{(1-\zeta_{2,eq}^{(1)})}$, $c = \frac{-M}{4N^{(2)}}$, $d = \frac{M}{2N^{(2)}}(3 - \zeta_{2,eq}^{(1)})$, $f = -\frac{c_2^{(1)}q_2^{(1)}}{P_a^{(1)}}$ and $e = -\frac{c_1^{(2)}q_1^{(2)}}{P_{eq}^{(2)}}$.

In Sec. III C, we have proved that the equilibrium point of system (21) are asymptotically stable if conditions **K1**-**K5** hold. Since system (40) is an example for system (21), the equilibrium point of system (40) is asymptotically stable if the same conditions hold.

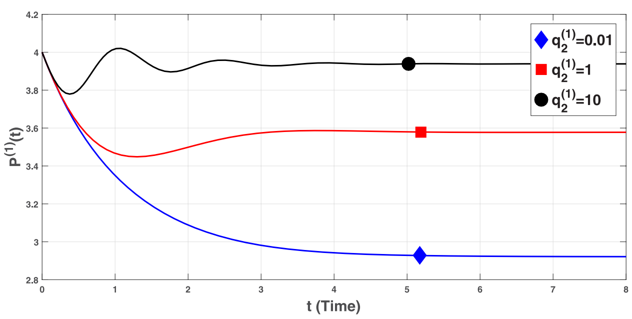

FIG. 1. Graphs of the first stock's price, $P^{(1)}(t)$, for $q_2^{(2)} = 1$ and $q_2^{(1)} = 0.01$, 1, and 10 marked with diamond, square, and circle, respectively. Here, $q_2^{(1)}$ is the valuation coefficient for the first stock, while $q_2^{(2)}$ is that for the second stock. We used $[P^{(1)}(0), P^{(2)}(0), \zeta_2^{(1)}(0), \zeta_2^{(2)}(0)] = (4, 6, 0.01, 0.01)$ as an initial condition for simulations. Note that the equilibrium price of the first stock, $P_{eq}^{(1)}$, is stable for each $q_2^{(1)}$. Moreover, it gets closer to the true value $P_a^{(1)} = 4$ as $q_2^{(1)}$ gets larger.

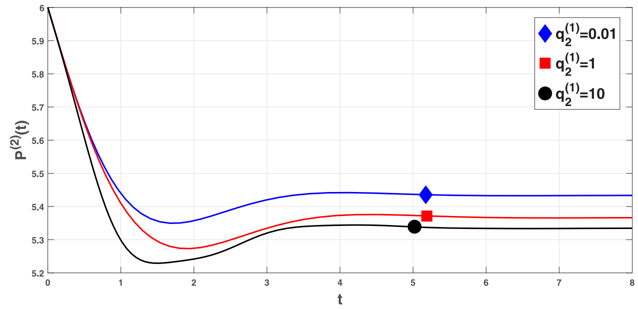

FIG. 2. Graphs of the price of the second stock, $P^{(2)}(t)$, for $q_2^{(2)} = 1$ and $q_2^{(1)} = 0.01$, 1, and 10 marked with diamond, square, and circle, respectively. We used the same initial condition as in Fig. 1. Simulations show that the steady state price of the second stock, $P_{eq}^{(2)}$, is stable for each $q_2^{(1)}$.

Since it is assumed that $\tanh(x) \simeq x$, we have $-1 < \zeta_2^{(1)} < 1$ so that one easily obtains the following inequalities:

$$0 < 1 - \zeta_2^{(1)} < 2, \tag{48}$$

$$0 < 1 + \zeta_2^{(1)} < 2, \tag{49}$$

$$2 < 3 - \zeta_2^{(1)} < 4. \tag{50}$$

By now using these inequalities, we show that condition **K1** holds as follows:

- Utilizing inequality (49), one has

$$\frac{\partial k^{(1)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) = -\frac{1}{8}[1 + \zeta_{2,eq}^{(1)}] < 0$$

and

$$\frac{\partial k^{(1)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) = \frac{1}{8}[3 - \zeta_{1,eq}^{(2)}] = \frac{3}{8} > 0,$$
$$\frac{\partial \tilde{k}^{(1)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) = -\frac{1}{2} < 0.$$

- Using now inequality (50), we can show that

TABLE I. The steady states of the stocks' prices in system (31) as $q_2^{(1)}$ varies.

$q_2^{(2)}$	$q_2^{(1)}$	$P_{eq}^{(1)}$	$P_{eq}^{(2)}$
1	0.01	2.9213	5.4335
1	1	3.5777	5.3666
1	10	3.9385	5.3346

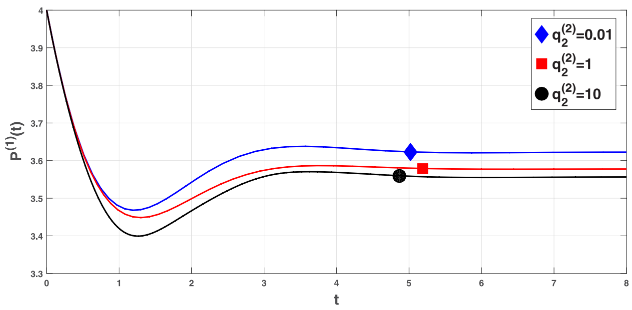

FIG. 3. Graphs of the price of the first stock, $P^{(1)}(t)$, for $q_2^{(1)} = 1$ and $q_2^{(2)} = 0.01, 1$, and 10 marked with diamond, square, and circle, respectively. Here, $q_2^{(1)}$ is the valuation coefficient for the first stock, while $q_2^{(2)}$ is that for the second stock. $[P^{(1)}(0), P^{(2)}(0), \zeta_2^{(1)}(0), \zeta_2^{(2)}(0)] = (4, 6, 0.01, 0.01)$ are the initial conditions used for simulations. It is clear that the equilibrium price of the first stock, $P_{eq}^{(1)}$, is stable for each $q_2^{(2)}$.

$$\frac{\partial k^{(2)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) = \frac{1}{8}[3 - \zeta_{2,eq}^{(1)}] > 0$$

and

$$\frac{\partial k^{(2)}}{\partial \zeta_2^{(1)}}(E_M^{eq}) = -\frac{1}{8}[1 + \zeta_{1,eq}^{(2)}] = -\frac{1}{8} < 0,$$

$$\frac{\partial \widetilde{k}^{(2)}}{\partial \zeta_1^{(2)}}(E_M^{eq}) = -\frac{1}{2} < 0.$$

Thus, condition **K1** holds. Then, according to Theorem 3, the equilibrium point of system (40) is asymptotically stable if

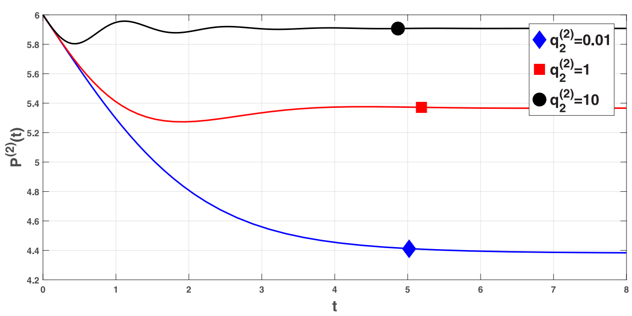

FIG. 4. Graphs of the second stock's price, $P^{(2)}(t)$, for $q_2^{(1)} = 1$ and $q_2^{(2)} = 0.01, 1$, and 10 marked with diamond, square, and circle, respectively. We used the same initial condition as in Fig. 3. Note that the equilibrium price of the second stock, $P_{eq}^{(2)}$, is stable for each $q_2^{(2)}$. Note also that it gets closer to the true value $P_a^{(2)} = 6$ as $q_2^{(2)}$ gets larger.

the following conditions hold:

$$\begin{aligned}
&\textbf{K2:}\quad 1 - 2q_1^{(2)} > 0,\\
&\textbf{K3:}\quad c_2^{(1)} + 1 - 2c_1^{(2)} > 0,\\
&\textbf{K4:}\quad 1 - \frac{3Mc_2^{(1)}q_2^{(1)}}{2N^{(1)}(1-\zeta_{2,eq}^{(1)})^2} > 0,\\
&\textbf{K5:}\quad 1 - \frac{M(1+\zeta_{2,eq}^{(1)})c_2^{(1)}q_2^{(1)}q_1^{(2)}}{4N^{(1)}(1-\zeta_{2,eq}^{(1)})(3-\zeta_{2,eq}^{(1)})P_a^{(1)}} > 0.
\end{aligned} \tag{51}$$

We summarize the result that we have concluded below.

Corollary 2. *The equilibrium point E_M^{eq} of system (40) is asymptotically stable if the conditions* **K2–K5** *in (51) hold.*

In plain language, Corollary 2 states that for our example the equilibrium is stable provided the following conditions are satisfied: (**K2**) the strength of the dependence of trend sentiment on stock 2 price should be less than 1/2, (**K3**) the trend sentiment for stock 2 must react slowly to the changes in the price of stock 2, and [(**K4**) and (**K5**)] the value sentiment for stock 1 must react slowly to the changes in the price of stock 1.

2. Bifurcation analysis for system (40)

In this subsection, we show the existence of the Hopf bifurcation of system (40) by choosing the trend coefficient of the second stock, $q_1^{(2)}$, as a bifurcation parameter. We first write the characteristic equation as follows:

$$H(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \tag{52}$$

where

$$a_1 = Aq_1^{(2)} + B, \tag{53}$$

$$a_2 = Fq_1^{(2)} + C, \tag{54}$$

$$a_3 = Gq_1^{(2)} + D, \tag{55}$$

$$a_4 = E \tag{56}$$

in which $A = -2c_1^{(2)}$, $B = 2 + c_1^{(2)} + c_2^{(1)}$, $C = 1 + 2c_1^{(2)} + 2c_2^{(1)} + c_1^{(2)}c_2^{(1)} + Kq_2^{(1)}c_2^{(1)}$, $D = c_1^{(2)} + c_2^{(1)} + 2c_1^{(2)}c_2^{(1)} + Kq_2^{(1)}c_2^{(1)}(1 + c_1^{(2)})$, $E = c_1^{(2)}c_2^{(1)} + Kq_2^{(1)}c_2^{(1)}c_1^{(2)}$, $F = -2c_1^{(2)}(1 + c_2^{(1)})$, $G = -2c_1^{(2)}(c_2^{(1)} + Kq_2^{(1)}c_2^{(1)}) + c_1^{(2)}c_2^{(1)}q_2^{(1)}L$, where

$$K = \frac{3M}{2N^{(1)}P_a^{(1)}(1 - \zeta_{2,eq}^{(1)})^2},$$

$$L = \frac{M^2(1 + \zeta_{2,eq}^{(1)})}{16N^{(1)}N^{(2)}P_a^{(1)}(1 - \zeta_{2,eq}^{(1)})}.$$

TABLE II. The steady states of the stocks' prices in system (31) as $q_2^{(2)}$ varies.

$q_2^{(1)}$	$q_2^{(2)}$	$P_{eq}^{(1)}$	$P_{eq}^{(2)}$
1	0.01	3.6223	4.3819
1	1	3.5777	5.3666
1	10	3.5564	5.9078

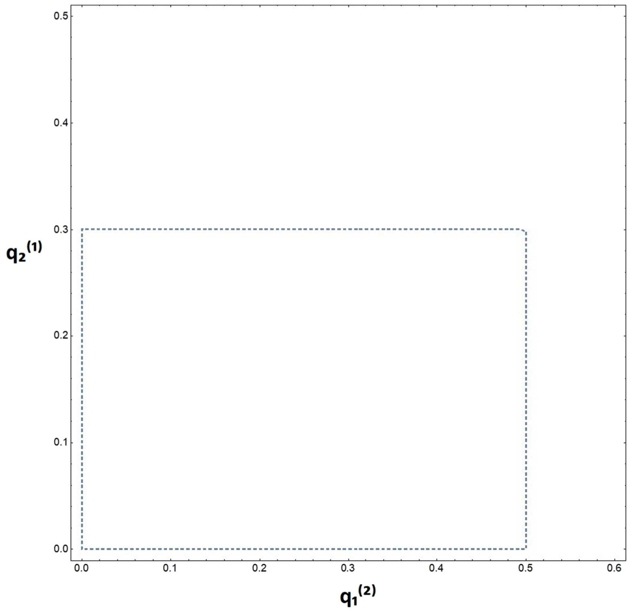

FIG. 5. The region bounded by the inner rectangle involves the values of $q_2^{(1)}$ and $q_1^{(2)}$ at which the steady states of system (40) are definitely stable.

Notice that since it is assumed that $\tanh(x) \simeq x$, one has $-1 < \zeta_2^{(1)} < 1$. Notice also that all parameters in K and L are positive. As a result, $K > 0$ and $L > 0$. Thus, $A < 0$, $B > 0$, $C > 0$, $D > 0$, $E > 0$, $F < 0$, and G can be either positive or negative.

The following theorem states the conditions on parameters at which system (40) has a Hopf bifurcation. We prove it using Theorem 6 proved by Asada and Yoshida (see Appendix D).

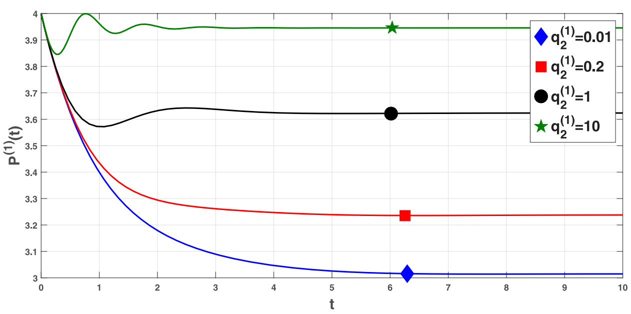

FIG. 6. Graphs of the price of the first stock, $P^{(1)}(t)$, for $q_1^{(2)} = 0.35$ and $q_2^{(1)} = 0.01, 0.2, 1,$ and 10, respectively, where $q_1^{(2)}$ is the trend coefficient for the second stock, while $q_2^{(1)}$ is the valuation coefficient for the first stock. The initial condition used for simulations is $[P^{(1)}(0), P^{(2)}(0), \zeta_2^{(1)}(0), \zeta_2^{(2)}(0)] = (4, 6, 0.01, 0.01)$. The equilibrium price of the first stock, $P_{eq}^{(1)}$, is stable for each $q_2^{(1)}$. Moreover, it gets closer to the true value $P_a^{(1)} = 4$ as $q_2^{(1)}$ gets larger.

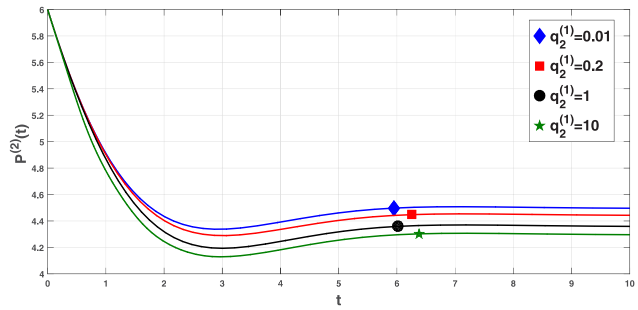

FIG. 7. Graphs of the second stock's price, $P^{(2)}(t)$, for $q_1^{(2)} = 0.35$ and $q_2^{(1)} = 0.01, 0.2, 1,$ and 10, respectively. $q_1^{(2)}$ is the trend coefficient for the second stock, while $q_2^{(1)}$ is the valuation coefficient for the first stock. We used the same initial conditions as in Fig. 6 for simulations. The equilibrium price of the second stock, $P_{eq}^{(2)}$, is stable for each $q_2^{(1)}$.

Lemma 1. *The characteristic polynomial $H(\lambda)$ has a pair of (simple) pure imaginary roots and two roots with negative real parts if one of the following condition holds:*

P1: $G < 0$ *and* $q_1^{(2)} < \min\left(-\frac{B}{A}, -\frac{D}{G}\right)$.

P2: $G \geq 0$ *and* $q_1^{(2)} < -(B/A)$.

Proof. To prove the claims, we utilize Theorem 6 which states that $H(\lambda)$ has a pair of pure imaginary roots and two roots with negative real parts iff $a_1 > 0$, $a_3 > 0$, $a_4 > 0$, and $\phi = a_1a_2a_3 - a_1^2a_4 - a_3^2 = 0$.

Let us first assume that **P1** is satisfied. Then, $a_3 = Gq_1^{(2)} + D$ is a linear decreasing function of $q_1^{(2)}$, because of $G < 0$. Similarly, $a_1 = Aq_1^{(2)} + B$ is a linear decreasing function of $q_1^{(2)}$ since $A < 0$. Also, $a_1 = 0$ and $a_3 = 0$ when $q_1^{(2)} = -\frac{B}{A}$ and $q_1^{(2)} = -\frac{D}{G}$, respectively, where $-\frac{B}{A} > 0$ and $-\frac{D}{G} > 0$. Thus, $a_1 > 0$ and $a_3 > 0$ for $q_1^{(2)} \in \left[0, \min\left(-\frac{B}{A}, -\frac{D}{G}\right)\right)$. We also know that $a_4 > 0$ due to its definition [see (56)].

On the other hand, ϕ is a continuous function of $q_1^{(2)}$ defined as follows:

$$\begin{cases} \phi(q_1^{(2)}) = a_1a_2a_3 - a_1^2a_4 - a_3^2 \\ \quad = AFG(q_1^{(2)})^3 + (CAG - EA^2 + FDA - F^2 + BCF)(q_1^{(2)})^2 \\ \quad +(ACD - 2ABE - 2GD + BFD + BCG)q_1^{(2)} \\ \quad +(CBD - EB^2 - D^2). \end{cases} \tag{57}$$

TABLE III. The steady states of the stocks' prices in system (40) as $q_2^{(1)}$ varies, but $q_1^{(2)}$ is fixed.

$q_2^{(1)}$	$q_1^{(2)}$	$P_{eq}^{(1)}$	$P_{eq}^{(2)}$
0.01	0.35	3.0148	4.4963
0.2	0.35	3.2377	4.4428
1	0.35	3.6235	4.3588
10	0.35	3.9455	4.2958

Using now the intermediate value theorem, we show that ϕ vanishes for some $q_1^{(2)} > 0$. Note that

$$\begin{cases} \phi(0) = \mathrm{CBD} - \mathrm{EB}^2 - \mathrm{D}^2 \\ \quad = \mathrm{K}^2 c_2^{(1)}(q_2^{(1)})^2 - 2\mathrm{K}c_2^{(1)}q_2^{(1)} + c_2^{(1)} \\ \quad\quad + \text{other positive terms} \\ \quad = c_2^{(1)}(\mathrm{K}q_2^{(1)} - 1)^2 + \text{other positive terms} \end{cases} \tag{58}$$

so that $\phi(0) > 0$.

Note also that $\phi(-\frac{B}{A}) = -(-\frac{GB}{A} + D)^2 < 0$ and $\phi(-\frac{D}{G}) = -(-\frac{AD}{G} + B)^2 E < 0$, so $\phi(q_1^{(2)}) < 0$ when $q_1^{(2)} = \min(-\frac{B}{A}, -\frac{D}{G})$. Then, by the intermediate value theorem, $\exists\, q_1^{(2),*} \in [0, \min(-\frac{B}{A}, -\frac{D}{G})]$ such that $\phi[q_1^{(2),*}] = 0$ [since ϕ is a polynomial of $q_1^{(2)}$]. Moreover, $a_1[q_1^{(2),*}] > 0$, $a_3[q_1^{(2),*}] > 0$.

Second, we assume that **P2** holds. Then, $G = 0$ implies $a_3 = D > 0$, and $G > 0$ implies $a_3 = Gq_1^{(2)} + D > 0$ for all $q_1^{(2)} > 0$. On the other hand, since $A < 0$, $B > 0$, and $a_1(-B/A) = 0$ [see (53)], $a_1 > 0$ when $q_1^{(2)} < -\frac{B}{A}$. Once again, $a_4 > 0$ by its definition. Finally, ϕ is a continuous function of $q_1^{(2)}$ [see (57)], $\phi(0) > 0$ [see (58)], and $\phi(-\frac{B}{A}) = -(-\frac{GB}{A} + D)^2 < 0$. By now using the intermediate value theorem, one can show that $\exists\, q_1^{(2),*} \in (0, -\frac{B}{A})$ such that $\phi(q_1^{(2),*}) = 0$.

Consequently, from Theorem 6 we have showed the existence of a pair of pure imaginary roots under conditions **P1** and **P2**. Moreover, Theorem 6 underlines that the pure imaginary roots are simple since the other two roots have negative real parts, and $H(\lambda)$ is a fourth order polynomial which has at most four zeros. □

Remark. *Since a pair of pure imaginary roots appearing when* $q_1^{(2)} = q_1^{(2),*}$ *is simple, the transversality condition holds, i.e.,*

$$\mathrm{Re}\left(\frac{d\lambda(q_1^{(2)})}{dq_1^{(2)}}\right)\Bigg|_{\{q_1^{(2),*};E_M^{eq}\}} \neq 0.$$

Theorem 4. *System (40) undergoes a Hopf bifurcation at* E_M^{eq} *if one of conditions* **P1** *and* **P2** *is satisfied.*

Proof. The proof follows from Lemma 1. □

V. NUMERICAL SIMULATIONS

In this section, we perform numerical simulations to support and extend the analytical results obtained in the former sections for the following two cases:

Case 1: All traders follow a fundamental strategy while selling or buying assets.

Case 2: The trading group follows a pure value-based strategy while selling or buying the first asset, and a pure trend-based strategy while selling or buying the second asset.

As a numerical example, we consider a closed market involving 2400 units of cash and 600 units of the first stock and 400 units of the second stock. We assume that the group values the first stock as $P_a^{(1)} = 4$ and the other stocks as $P_a^{(2)} = 6$. For each simulation, we use the ODE package (ode23s) in MATLAB (R2016a).

Case 1: In this case, we fixed time scales for the valuation motivations as $c_2^{(1)} = 1$ and $c_2^{(2)} = 1$. Corollary 1 underlines that each equilibrium is asymptotically stable for all positive parameters. In Figs. 1 and 2, we fix magnitude for the valuation

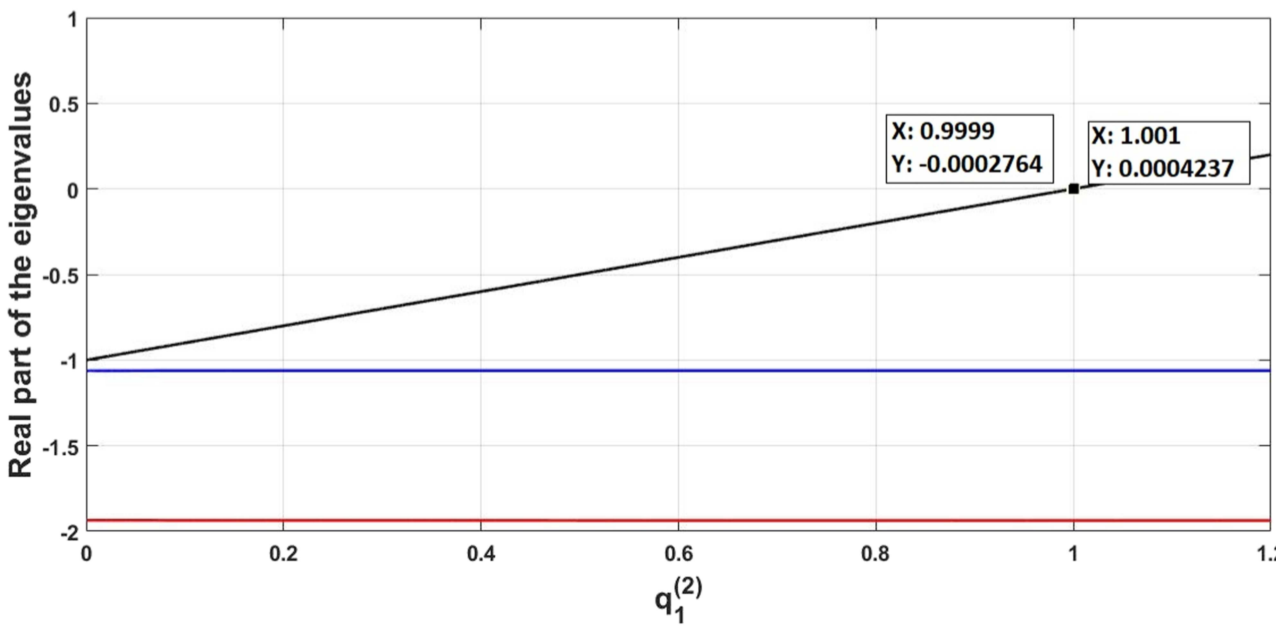

FIG. 8. Graphs of the real parts of eigenvalues of the Jacobian matrix of system (40) versus $q_1^{(2)}$. In this graph, X:=$q_1^{(2)}$ and Y:=Real part of an eigenvalue.

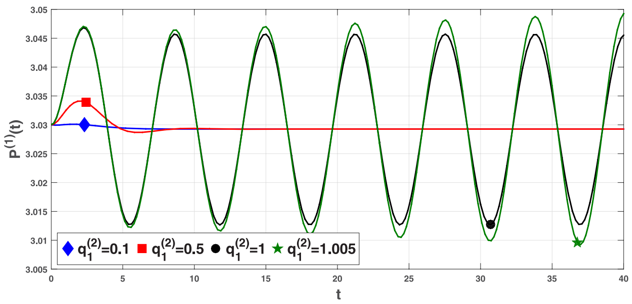

FIG. 9. Graphs of the price of the first stock, $P^{(1)}(t)$, for $q_2^{(1)} = 0.02$ and $q_1^{(2)} = 0.1, 0.5, 1$, and 1.005, respectively. $[P^{(1)}(0), P^{(2)}(0), \zeta_2^{(1)}(0), \zeta_1^{(2)}(0)] = (3.03, 4.6, 0.0049, 0)$ is the set of the initial conditions used for simulations. The price is stable for $q_1^{(2)} = 0.1$ and $q_1^{(2)} = 0.5$, unstable for $q_1^{(2)} = 1.005$, periodic for $q_1^{(2)} = 1 \doteq q_1^{(2),*}$.

motivation for the second stock as $q_2^{(2)} = 1$ and vary the value-based coefficient $q_2^{(1)}$ for the first stock. We take $\mathrm{P}^{(1)}(0) = 4$, $\mathrm{P}^{(2)}(0) = 6$, $\zeta_2^{(1)}(0) = 0.01$, and $\zeta_2^{(2)}(0) = 0.01$ as an initial condition and plot graphs of solutions for the stocks' prices by using the parameters above. The equilibrium prices in Table I vary with $q_2^{(1)}$ (see Appendix B which explains the calculations of the equilibrium points $\mathrm{P}_{eq}^{(1)}$ and $\mathrm{P}_{eq}^{(2)}$).

Figures 1 and 2 show that the equilibrium point is stable for each value of $q_2^{(1)}$ which is compatible with Corollary 1. From these figures, one can also observe that $\mathrm{P}_{eq}^{(1)}$ tends to $\mathrm{P}_a^{(1)}$ while $q_2^{(1)}$ gets larger. This means that if the investor group pays more attention to the valuation of the first stock, then the equilibrium price of the first stock gets close to its fundamental value that yields a similar conclusion obtained in Ref. 14.

In Figs. 3 and 4, we now fix the magnitude for the valuation motivation for the first stock as $q_2^{(1)} = 1$ and vary the value-based coefficient for the second stock as $q_2^{(2)} = 0.01$, 1, 10, respectively. By using the parameters values used in Figs. 1 and 2 together with the initial conditions $[\mathrm{P}^{(1)}(0), \mathrm{P}^{(2)}(0), \zeta_2^{(1)}(0), \zeta_2^{(2)}(0)] = (4, 6, 0.01, 0.01)$, we plot graphs of solutions for the stocks' prices once again. The equilibrium prices, which vary with respect to the parameter $q_2^{(2)}$, are given in Table II (see Appendix B for their calculations). These graphs show that the equilibrium point is stable for each given value of $q_2^{(2)}$. According to Fig. 4, while $q_2^{(2)}$ gets larger, the equilibrium price of the second stock gets close to $\mathrm{P}_a^{(2)}$, which is the fundamental value for the second stock.

Case 2. We again fix time scales for both trend and valuation motivations as $c_2^{(1)} = 2$ and $c_1^{(2)} = 1$. First, using the criteria given by Corollary 2, we display the stability region where all fixed points are stable as the parameters $q_2^{(1)}$ and $q_1^{(2)}$ vary within. For the parameters which are located inside of this region, the steady states of system (40) must be stable. However, for the parameters which are located outside or on the boundary of this region, the corresponding equilibrium may be either stable or unstable.

In Figs. 6 and 7, we fix the magnitude for trend motivations as $q_1^{(2)} = 0.35$ and vary valuation motivations as $q_2^{(1)} = 0.01, 0.2, 1$, and 10, respectively [note that $(q_1^{(2)}, q_2^{(1)}) = (0.35, 0.01)$ and $(q_1^{(2)}, q_2^{(1)}) = (0.35, 0.2)$ are located inside of the stability region, but $(q_1^{(2)}, q_2^{(1)}) = (0.35, 1)$ and $(q_1^{(2)}, q_2^{(1)}) = (0.35, 10)$ are located outside of the stability region exhibited in Fig. 5]. Using these parameters, we plot the graphs of the first two components (stocks' prices) of the solutions by taking $[\mathrm{P}^{(1)}(0), \mathrm{P}^{(2)}(0), \zeta_2^{(1)}(0), \zeta_2^{(2)}(0)] = (4, 6, 0.01, 0.01)$ as an initial condition. According to the parameters, for each value of $q_2^{(1)}$, system (40) has only one positive equilibrium point that is given by Eqs. (46) and (47). The equilibrium prices are given in Table III. Figures 6 and 7 illustrate that the equilibrium prices that vary with respect to $q_2^{(1)}$ are stable.

In Fig. 8, fixing the magnitude of the valuation motivation as $q_2^{(1)} = 0.02$ and using the parameters above, we plot the real parts of the eigenvalues of the Jacobian matrix of system (40) according to the trend coefficient for the second stock, $q_1^{(2)}$. With respect to Eq. (46), $\mathrm{P}_{eq}^{(1)}$ and $\mathrm{P}_{eq}^{(2)}$ are independent of $q_1^{(2)}$, so the system has a unique equilibrium value $[\mathrm{P}_{eq}^{(1)}, \mathrm{P}_{eq}^{(2)}, \zeta_{2,eq}^{(1)}, \zeta_{1,eq}^{(2)}] = (3.0293, 4.4927, 0.0049, 0)$. This graph shows that the equilibrium point of the system is unstable for each $q_1^{(2)}$ which is bigger than a critical value $q_1^{(2),*} \in (0.9999, 1.001)$ since the real part of one of the eigenvalues is positive for $q_1^{(2)} > q_1^{(2),*}$.

Figures 9 and 10 present the graphs of the stocks' prices when $q_1^{(2)} = 0.1, 0.5, 1$, and 1.005, respectively, for the fixed valuation coefficient, namely, $q_2^{(1)} = 0.02$. These graphs are plotted by taking the initial conditions as $\mathrm{P}^{(1)}(0) = 3.03$, $\mathrm{P}^{(2)}(0) = 4.6$, $\zeta_2^{(1)}(0) = 0.0049$, $\zeta_1^{(2)}(0) = 0$. Note again that except for $[q_1^{(2)}, q_2^{(1)}] = (0.01, 0.02)$ and $[q_1^{(2)}, q_2^{(1)}] = (0.5, 0.02)$, the other points are located outside of the stability region in Fig. 5. According to Figs. 9 and 10, the equilibrium prices are stable for $q_1^{(2)} = 0.1$ and $q_1^{(2)} = 0.5$, while they are unstable for $q_1^{(2)} = 1.005$ which is bigger than $q_1^{(2),*}$. Moreover, when $q_1^{(2)} = 1$ that is very closed to the critical value $q_1^{(2),*}$, the system presents a cyclic behavior which underlines the existence of periodic solutions through a Hopf bifurcation as $q_1^{(2)}$ passes

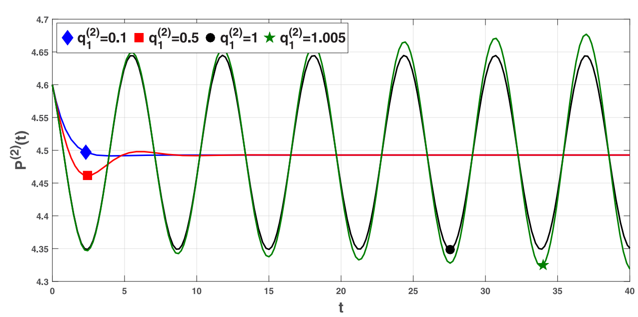

FIG. 10. Graphs of the second stock's price, $P^{(2)}(t)$, for $q_2^{(1)} = 0.02$ and $q_1^{(2)} = 0.1, 0.5, 1$, and 1.005, respectively. We used the same initial conditions and observed a similar behavior as in Fig. 9.

through $q_1^{(2),*}$ by choosing $q_1^{(2)}$ as a bifurcation parameter (see Theorem 4). Next few simulations show this behavior.

Finally, we perform numerical simulations to support and extend the existence of the Hopf bifurcation of system (40). For simulations, we again fix the value-based coefficient of the first stock as $q_2^{(1)} = 0.02$. Using the parameters $c_2^{(1)} = 2$, $c_1^{(2)} = 1$, $P_a^{(1)} = 4$, and $P_a^{(2)} = 6$, we calculate G, $-\frac{B}{A}$, and $-\frac{D}{G}$ as follows:

$$G = -4.1514, \quad -\frac{B}{A} = 2.5000, \text{ and } -\frac{D}{G} = 1.7153.$$

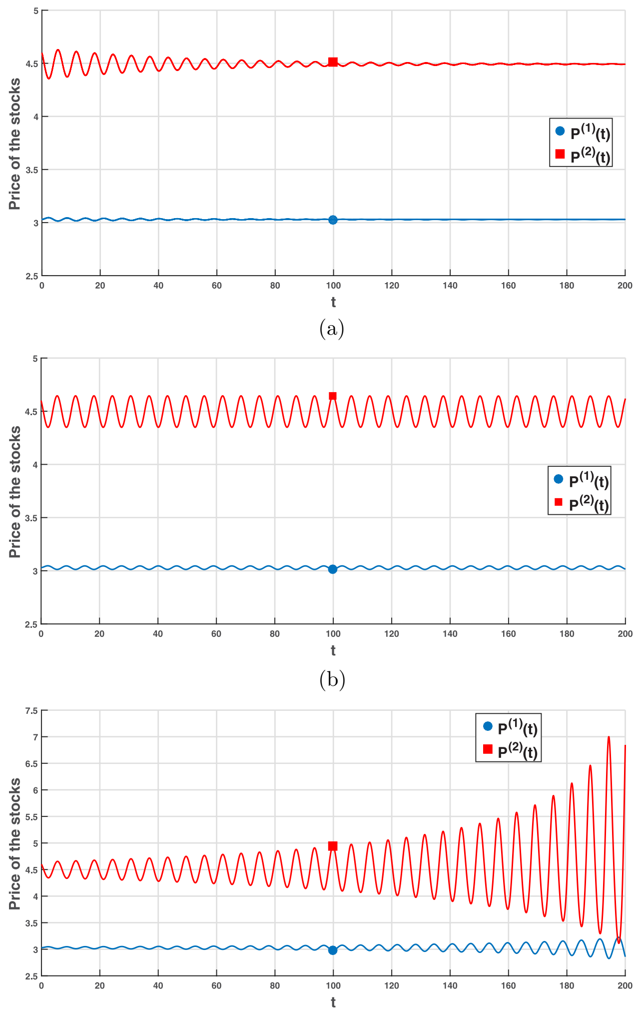

FIG. 11. Graphs of the stocks' prices, $P^{(1)}(t)$ and $P^{(2)}(t)$, for $q_1^{(2)} = 0.98 < q_1^{(2),*}$ (a), $q_1^{(2)} = 1 \doteq q_1^{(2),*}$ (b), and $q_1^{(2)} = 1.01 > q_1^{(2),*}$ (c). $[P^{(1)}(0), P^{(2)}(0), \zeta_2^{(1)}(0), \zeta_1^{(2)}(0)] = (3.03, 4.6, 0.0049, 0)$ is the set of the initial conditions used for these simulations. The steady state prices are stable in (a), periodic in (b), and unstable in (c).

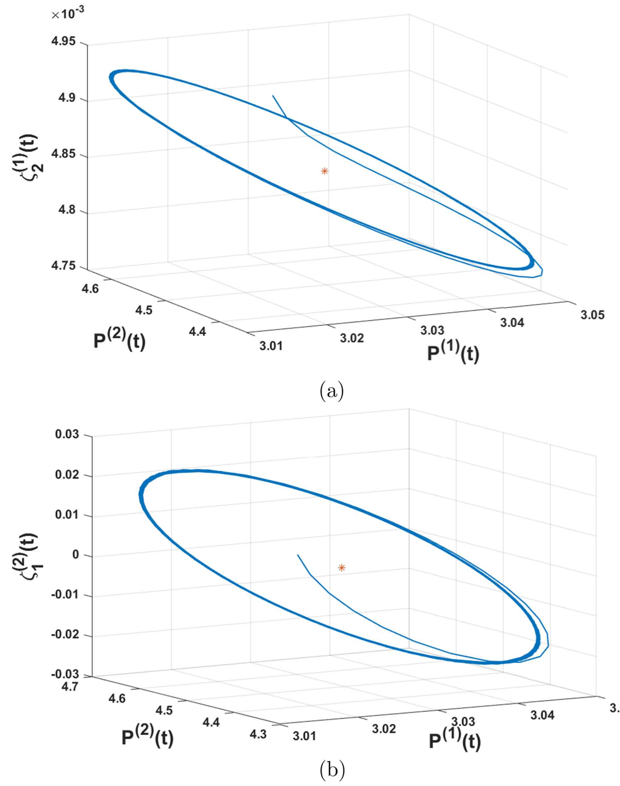

FIG. 12. Graphs of the trajectories of the solutions for $[P^{(1)}(t), P^{(2)}(t), \zeta_2^{(1)}(t)]$ (a) and $[P^{(1)}(t), P^{(2)}(t), \zeta_1^{(2)}(t)]$ (b) when $q_1^{(2)} = 1 \doteq q_1^{(2),*}$. We used the same initial conditions as in Fig. 11 for these simulations. The star denotes the equilibrium point.

Hence, according to condition **P1** in Theorem 4, the Hopf bifurcation occurs at $q_1^{(2),*} \doteq 1 < \min\left(-\frac{B}{A}, -\frac{D}{G}\right)$, where $q_1^{(2),*}$ is the root of the function $\phi(q_1^{(2)})$ [see (57)] (note here that if ϕ has two roots or more, then the smallest positive root should be chosen as $q_1^{(2),*}$).

In Fig. 11, we plot the stocks' prices, $P^{(1)}(t)$ and $P^{(2)}(t)$, according to time for $q_1^{(2)} = 0.98 < q_1^{(2),*}$ in (a), $q_1^{(2)} = 1 \doteq q_1^{(2),*}$ in (b), and $q_1^{(2)} = 1.01 > q_1^{(2),*}$ in (c). The initial conditions that we used for simulations are $P^{(1)}(0) = 3.03$, $P^{(2)}(0) = 4.6$, $\zeta_2^{(1)}(0) = 0.0049$, and $\zeta_1^{(2)}(0) = 0$. The graphs in (a) show that the equilibrium points of the stocks' prices are stable for $q_1^{(2)}$ which is less than the critical bifurcation value, $q_1^{(2),*}$. The graphs in (b) illustrate that periodic solutions occur through a Hopf bifurcation as the bifurcation parameter, $q_1^{(2)}$, passes through $q_1^{(2),*}$. The graphs in (c) indicate that the steady state prices are unstable for $q_1^{(2)}$ that is greater than $q_1^{(2),*}$. Furthermore, Fig. 12 presents the trajectories of the solutions for $[P^{(1)}(t), P^{(2)}(t), \zeta_2^{(1)}(t)]$ [in (a)] and $[P^{(1)}(t), P^{(2)}(t), \zeta_1^{(2)}(t)]$ [in (b)] for $q_1^{(2)} = 1 \doteq q_1^{(2),*}$.

Finally, Fig. 13 illustrates the trajectories of the solutions for $[P^{(1)}(t), P^{(2)}(t), \zeta_2^{(1)}(t)]$ [in (a)] and $[P^{(1)}(t), P^{(2)}(t), \zeta_1^{(2)}(t)]$

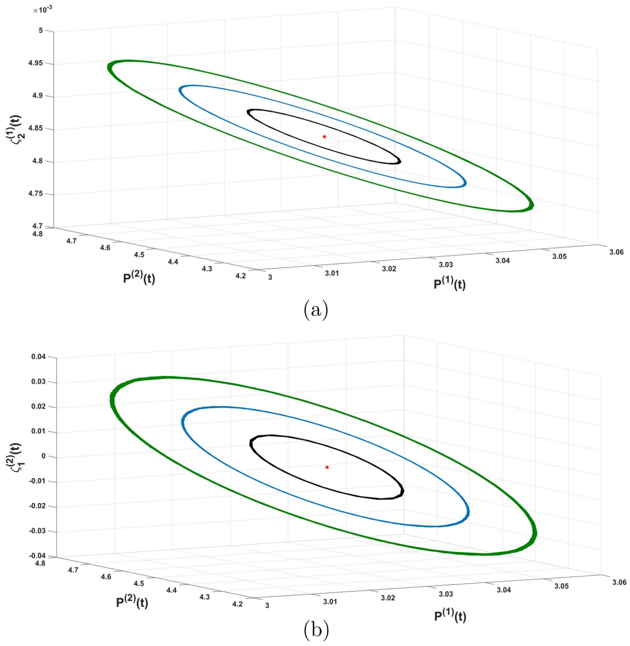

FIG. 13. Graphs of the trajectories of $[P^{(1)}(t), P^{(2)}(t), \zeta_2^{(1)}(t)]$ (a) and $[P^{(1)}(t), P^{(2)}(t), \zeta_1^{(2)}(t)]$ (b) for $q_1^{(2)} = 1 \doteq q_1^{(2),*}$ with the different initial values: $P^{(2)}(0) = 4.55$ (inner), $P^{(2)}(0) = 4.6$ (middle), and $P^{(2)}(0) = 4.65$ (outer). The star denotes the equilibrium point. This figure presents the segment of the trajectories for $t \in [20; 200]$.

[in (b)] with different initial values. For these simulations, we take $q_1^{(2)} = q_1^{(2),*}$, but use the different initial values, namely, $[P^{(1)}(0), P^{(2)}(0), \zeta_2^{(1)}(0), \zeta_1^{(2)}(0)] = (3.03, 4.55, 0.0049, 0)$ (inner), $[P^{(1)}(0), P^{(2)}(0), \zeta_2^{(1)}(0), \zeta_1^{(2)}(0)] = (3.03, 4.6, 0.0049, 0)$ (middle), and $[P^{(1)}(0), P^{(2)}(0), \zeta_2^{(1)}(0), \zeta_1^{(2)}(0)] = (3.03, 4.65, 0.0049, 0)$ (outer). The periods of the periodic solutions increase. Figure 13 shows the segment of the trajectories for $t \in [20; 200]$.

In numerical analysis of Hopf bifurcations, it is customary to add the bifurcation diagram which indicates the magnitude of the limit cycle that usually accompanies such a bifurcation. Unfortunately, for the example we study here, limit cycles do not appear on neither side of the bifurcation, and hence, the bifurcation diagram is trivial. Instead of converging to a limit cycle, the solutions depart the unstable equilibrium and after several oscillations they blow up (diverge to infinity) at a finite time. We have verified this observation using extensive numerical simulations and the bifurcation analysis software XPPAUT. The peculiarity of this non-generic behavior, which occurs for both the linearized and the nonlinear version of the example, lies in the specific assumptions on the transition rate functions we made in (10) and (39), in particular, the forms of $\widetilde{k}^{(1)}$ and $\widetilde{k}^{(2)}$. When these forms are replaced by constant functions, i.e., $\widetilde{k}^{(1)} = \frac{1}{2}$ and $\widetilde{k}^{(2)} = \frac{1}{2}$, one can easily observe stable limit cycle behavior (not shown).

VI. CONCLUSIONS

Since early 1990s, the dynamics of asset prices has been studied through a dynamical system approach by considering a market that consists of a single asset and a group of investors and utilizing several key aspects that are often followed in practice.[6,14–18,21–23,25–27] Unlike the assumptions of neoclassical economics that are largely based on the efficient market hypothesis, these studies assume the finiteness of assets, which ignores the arbitrage argument and also different motivations and strategies in the trading that eliminate the unique price argument. In each paper, the starting point is the excess demand equation which is the basic principle of economics given by Eq. (2).

In this paper, we study asset price dynamics of a market involving two assets and a group of investors who have common motivations and strategies for trading. It is assumed that the stocks are distributed to this single homogeneous group randomly, and the group follows a trading strategy in which the buying of an asset depends on the other asset's price while the selling does not. Utilizing the basic microeconomics principle, we derive a mathematical model which is based on this trading strategy together with the idea of the finiteness of assets and preference that is influenced by price momentum and discount from fundamental value. This model differs from the former asset flow models that argue a single asset market system.[14–18,25]

We have performed the stability analysis of the model and determined the conditions on parameters that guarantee stability. First, we showed that if the group of investors focuses on fundamental values of each stock for trading, then all equilibria are stable provided trading rates depend in a reasonable fashion on value sentiments, in particular, when the influence of a stock sentiment on its own trading rate is larger than its influence on the trading rate of the other stock. Second, we established conditions for stability for the system in which the investor group pays attention to both the valuation of stock 1 and the trend of stock 2 when trading them. The most significant finding here is that, similar to the results obtained for a single-stock market, stability requires that both the trend and value sentiments react slowly to price variations and that the trend sentiment's dependence on price growth and decline is small. Then, a Hopf bifurcation analysis is given for the latter case that leads to the cyclic behavior for such market systems under the emphasis of the strong momentum effects. Finally, analytical results have been supported and extended by numerical simulations. Numerical studies show that an equilibrium that is stable becomes unstable as the trend based trading increases, and a Hopf bifurcation occurs as the trend based coefficient of stock 2, $q_1^{(2)}$, passes through a critical value (see, for example, Fig. 8). This result is important from the market point of view since the classical finance always treats the equilibrium point as a single point.[1–3,19,34]

Advantages of the asset flow models with respect to the classical finance models of asset price dynamics include the following: (i) In the classical models prices behave according to the Brownian motion, so it is not possible to analyze market microstructure and investors' strategies.[5,7–9,11–13] However, since the asset flow approach contains the relatively simple assumption arising in practical applications, these topics can be studied by using the tools of differential equations.[6,14,16,21,25,27] (ii) The classical models often argue a unique price since the arbitrage argument is generally assumed. As a result, they exclude any periodic behavior and treat any large, rapid deviation in price (often called as "market crash") as a rare probabilistic event.[2,3,19,24,34] However, stability, instability, and cyclic behavior in an asset market can be analyzed by utilizing the asset flow models.[6,14,15,21,23–25,27] From this perspective, the model introduced here has a potential to study a variety of issues arising in the financial market, such as the qualitative price behavior in an asset market[16,18,25] and dynamics of the market presenting speculative bubbles or major crashes, which methods of classical finance fail to explain.[10,13,24,27,28,30] An extension of this model for a market system involving multiple stocks and heterogeneous investors groups may further improve the understanding of dynamics of financial markets. These will be topics for a future study.

ACKNOWLEDGMENTS

We would like to express our gratitude to Dr. Gunduz Caginalp for his valuable discussion, comments, and suggestions that led to truly significant improvement of this manuscript. H. Merdan was supported by TUBITAK (The Scientific and Technological Research Council of Turkey).

APPENDIX A: ROUTH-HURWITZ CRITERIA

Theorem 5. *Given the polynomial*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n,$$

where the coefficients a_i are real constants, $i = 1, \ldots, n$, define the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial as follows:

$$H_1 = (a_1),\ H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix},\ H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

where $a_j = 0$, if $j > n$. Then, all of the roots of the polynomial $P(\lambda)$ are negative or have a a negative real part if and only if the determinants of all Hurwitz matrices are positive, i.e.,

$$\det H_j > 0, \quad j = 1, 2, \ldots, n.$$

The Routh-Hurwitz criteria for $n = 3, 4$ then have the following conditions:

$$n = 3 : a_1 > 0, a_3 > 0,\ \textit{and}\ a_1a_2 > a_3,$$

$$n = 4 : a_1 > 0, a_3 > 0, a_4 > 0,\ \textit{and}\ a_1a_2a_3 > a_3^2 + a_1^2a_4.$$

APPENDIX B: EQUILIBRIUM POINTS OF SYSTEM (31)

The equilibrium points of system (31) are obtained by equating the right hand sides of the equations in system to 0. Equating the first two equations to zero yields

$$P_{eq}^{(1)} = \frac{(1+\zeta_{2,eq}^{(1)})(3-\zeta_{2,eq}^{(2)})M}{4(1-\zeta_{2,eq}^{(1)})N^{(1)}}, \tag{B1}$$

$$P_{eq}^{(2)} = \frac{(1+\zeta_{2,eq}^{(2)})(3-\zeta_{2,eq}^{(1)})M}{4(1-\zeta_{2,eq}^{(2)})N^{(2)}}. \tag{B2}$$

From the last two equations in system and Eqs. (B1) and (B2), we then have the following nonlinear equations:

$$G(\zeta_{2,eq}^{(1)}, \zeta_{2,eq}^{(2)}) = c_2^{(1)}q_2^{(1)}\left(1 - \frac{M(1+\zeta_{2,eq}^{(1)})(3-\zeta_{2,eq}^{(2)})}{4N^{(1)}P_a^{(1)}(1-\zeta_{2,eq}^{(1)})}\right) - c_2^{(1)}\zeta_{2,eq}^{(1)},$$

$$H(\zeta_{2,eq}^{(1)}, \zeta_{2,eq}^{(2)}) = c_2^{(2)}q_2^{(2)}\left(1 - M\frac{(1+\zeta_{2,eq}^{(2)})(3-\zeta_{2,eq}^{(1)})}{4N^{(2)}P_a^{(2)}(1-\zeta_{2,eq}^{(2)})}\right) - c_2^{(2)}\zeta_{2,eq}^{(2)}.$$

In Sec. V, to find the equilibrium points of system 31, numerically we first fixed all parameters ($c_2^{(1)}$, $c_2^{(2)}$, $q_2^{(1)}$, $q_2^{(2)}$, M, $N^{(1)}$, $N^{(2)}$, $P_a^{(1)}$, $P_a^{(2)}$), and then by using "fsolve" function in MATLAB, we find the root of the following system:

$$\begin{cases} G(\zeta_{2,eq}^{(1)}, \zeta_{2,eq}^{(2)}) = 0, \\ H(\zeta_{2,eq}^{(1)}, \zeta_{2,eq}^{(2)}) = 0. \end{cases} \tag{B3}$$

It is also possible to find the root by using Newton's method.

APPENDIX C: THE DISTRIBUTION OF WEALTH

Notice that for each asset, we have

$$\frac{1}{P^{(i)}}\frac{dP^{(i)}}{dt} = \frac{k^{(i)}M}{\tilde{k}^{(i)}N^{(i)}P^{(i)}} - 1$$

[see Eqs. (8) and (9) with $F_i(x) = x - 1$] so that in equilibrium

$$P^{(i)} = \frac{k^{(i)}M}{\tilde{k}^{(i)}N^{(i)}}.$$

The fraction of assets in cash is then

$$\begin{aligned} W_C(k^{(1)}, k^{(2)}, \tilde{k}^{(1)}, \tilde{k}^{(2)}) &:= \frac{M}{M + N^{(1)}P^{(1)} + N^{(2)}P^{(2)}} \\ &= \frac{M}{M + \frac{k^{(1)}}{\tilde{k}^{(1)}}M + \frac{k^{(2)}}{\tilde{k}^{(2)}}M} \\ &= \frac{1}{1 + \frac{k^{(1)}}{\tilde{k}^{(1)}} + \frac{k^{(2)}}{\tilde{k}^{(2)}}}. \end{aligned}$$

Similarly, the fractions of the total wealth in stock 1 and stock 2 are

$$W_1(k^{(1)}, k^{(2)}, \widetilde{k}^{(1)}, \widetilde{k}^{(2)}) := \frac{\frac{k^{(1)}}{\widetilde{k}^{(1)}}}{1 + \frac{k^{(1)}}{\widetilde{k}^{(1)}} + \frac{k^{(2)}}{\widetilde{k}^{(2)}}},$$

$$W_2(k^{(1)}, k^{(2)}, \widetilde{k}^{(1)}, \widetilde{k}^{(2)}) := \frac{\frac{k^{(2)}}{\widetilde{k}^{(2)}}}{1 + \frac{k^{(1)}}{\widetilde{k}^{(1)}} + \frac{k^{(2)}}{\widetilde{k}^{(2)}}}.$$

APPENDIX D: THE ROOTS OF THE FOURTH DEGREE POLYNOMIALS

The following theorem was proved by Asada and Yoshida.[35]

Theorem 6. *(i) The polynomial equation*

$$\delta(\lambda) = \lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0 \tag{D1}$$

has a pair of pure imaginary roots and two roots with non-zero real parts if and only if either of the following set of conditions **(A)** *or* **(B)** *is satisfied:*

(A) $b_1b_3 > 0, b_4 \neq 0,$ *and* $\phi \equiv b_1b_2b_3 - b_1^2b_4 - b_3^2 = 0.$
(B) $b_1 = 0, b_3 = 0,$ *and* $b_4 < 0.$

(ii) The polynomial equation (D1) has a pair of pure imaginary roots and two roots with negative real parts if and only if the following set of conditions **(C)** *is satisfied:*

(C) $b_1 > 0, b_3 > 0, b_4 > 0,$ *and* $\phi \equiv b_1b_2b_3 - b_1^2b_4 - b_3^2 = 0.$

REFERENCES

[1]Z. Bodie, A. Kane, and A. J. Marcus, *Investments*, 7th ed. (McGraw-Hill Education, New York, 2008).
[2]D. Fudenberg and J. Tirole, *Game Theory* (Massachusetts Institute of Technology, 1991).
[3]J. M. Henderson and R. E. Quant, *Microeconomic Theory: A Mathematical Approach* (McGraw-Hill, 1980).
[4]P. Wilmott, *Paul Wilmott Introduces Quantitative Finance* (John Wiley & Sons, 2007).
[5]T. Beard and B. Beil, "Do people rely on the self-interested maximization of others? An experimental test," Manage. Sci. **40**, 252 (1994).
[6]G. Caginalp and H. Merdan, "Asset price dynamics with heterogenous groups," Physica D **225**, 43 (2007).
[7]K. D. Daniel, D. Hirshleifer, and A. Subrahmanyam, "Investor psychology and security market under and overreaction," J. Finance **53**, 1839 (1998).
[8]D. Davis and C. Holt, *Experimental Economics* (Princeton University Press, New Jersey, 1993).
[9]L. Lopes, "Between hope and fear: The psychology of risk," Adv. Exp. Soc. Psychol. **20**, 255 (1987).
[10]J. M. Poterba and L. H. Summers, "Mean reversion in stock prices: Evidence and implications," J. Financ. Econ. **22**, 27 (1988).
[11]H. Shefrin, *A Behavioral Approach to Asset Pricing* (Elsevier, New York, 2005).
[12]H. Shefrin and M. Statman, "The disposition to sell winners too early and ride losers too long: Theory and evidence," J. Finance **40**, 777 (1985).
[13]V. L. Smith, G. L. Suchanek, and A. W. Williams, "Bubbles, crashes and endogenous expectations in experimental spot asset markets," Econometrica **56**, 1119 (1988).
[14]G. Caginalp and D. Balenovich, "Asset flow and momentum: Deterministic and stochastic equations," Phil. Trans. R. Soc. Lond. A **357**, 2119 (1999).
[15]H. Merdan, G. Caginalp, and W. Troy, "Bifurcation analysis of a single-group asset flow model," Q. Appl. Math. **74**, 275 (2016).
[16]G. Caginalp and G. B. Ermentrout, "A kinetic thermodynamics approach to the psychology of fluctuations in financial markets," Appl. Math. Lett. **3**, 17 (1990).
[17]G. Caginalp and G. B. Ermentrout, "Numerical studies of differential equation related to theoretical financial markets," Appl. Math. Lett. **4**, 35 (1991).
[18]G. Caginalp and D. Balenovich, "Market oscillations induced by the competition between value-based and trend-based investment strategies," Appl. Math. Finance **1**, 29 (1994).
[19]D. S. Watson and M. Getz, *Price Theory and Its Uses* (University Press of America, Lanham, MD, 1993).
[20]G. Caginalp, D. Porter, and V. L. Smith, "Initial cash/asset ratio and asset prices: An experimental study," Proc. Natl. Acad. Sci. **95**, 756 (1998).
[21]G. Caginalp and M. DeSantis, "Multi-group asset flow equations and stability," Discrete Cont. Dyn. Syst. Ser. B **16**109 (2011).
[22]G. Caginalp and M. DeSantis, "Nonlinearity in the dynamics of financial markets," Nonlinear Anal. Real World Appl. **12**, 1140 (2011).
[23]G. Caginalp and M. DeSantis, "Stock price dynamics: Nonlinear trend, volume, volatility, resistance and money supply," Quant. Finance **11**, 849 (2011).
[24]G. Caginalp, M. DeSantis, and D. Swigon, "Are flash crashes caused by instabilities arising from rapid trading?," Wilmott Mag. **11**, 46 (2011).
[25]H. Merdan and M. Alisen, "A mathematical model for asset pricing," Appl. Math. Comput. **218**, 1449 (2011).
[26]M. DeSantis, D. Swigon, and G. Caginalp, "Nonlinear dynamics and stability in a multi-group asset flow model," SIAM J. Appl. Dyn. Syst. **11**, 1114 (2012).
[27]H. Merdan and H. Cakmak, "Liquidity effect on the asset price forecasting," J. Nonlinear Syst. Appl. **3**, 82 (2012).
[28]G. Caginalp, D. Porter, and V. L. Smith, "Momentum and overreaction in experimental asset markets," Int. J. Ind. Organ. **18**, 187 (2000).
[29]G. Caginalp and V. Ilieva, "Hybrid methodology for technical analysis," Nonlinear Anal. Hybrid Syst. **2**, 1144 (2008).
[30]G. Caginalp and V. Ilieva, "The dynamics of trader motivations in asset bubbles," J. Econ. Behav. Organ. **66**, 6412008).
[31]M. DeSantis and D. Swigon, "Slow-fast analysis of a multi-group asset flow model with implications for the dynamics of wealth," PLoS ONE **13**, e0207764 (2018).
[32]L. J. S. Allen, *An Introduction to Mathematical Biology* (Pearson Education, Upper Saddle River, NJ, 2007).
[33]J. Hale and H. Kocak, *Dynamics and Bifurcations* (Springer-Verlag, 1991).
[34]J. Quirk and R. Rupert, "Qualitative economics and the stability of equilibrium," Rev. Econ. Stud. **32**, 311 (1965).
[35]T. Asada and H. Yoshida, "Coefficient criterion for four-dimensional Hopf bifurcations: A complete mathematical characterization and applications to economic dynamics," Chaos Solitons Fractals **18**, 525 (2003).